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State Reunion Maintainability for Semi-Markov Models

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Abstract

In previous research the importance of both Markov and semi-Markov models in manpower planning is highlighted. Maintainability of population structures for different types of personnel strategies (i.e. under control by promotion and control by recruitment) were extensively investigated for various types of Markov models (homogeneous as well as non-homogeneous) ([Bartholomew, 1967], [Vassiliou and Tsantas, 1984a]). Semi-Markov models are extensions of Markov models that account for duration of stay in the states. Less attention is paid to the study of maintainability for semi-Markov models. Although, some interesting maintainability results were obtained for non-homogeneous semi-Markov models ([Vassiliou and Papadopoulou, 1992]).

The current paper focuses on discrete-time homogeneous semi-Markov models, and explores the concept of maintainable population structures in this setting. In particular, a new concept of maintainability is introduced, the so-called State Reunion maintainability with respect to a matrix $\mathbf{M}$ (SR_M-maintainability). It is shown that this concept of maintainability is closely related to maintainability for non-homogeneous Markov chains.

Keywords: Markov model, semi-Markov model, maintainability, state reunion maintainability, control theory, manpower planning, non-homogeneous Markov chain

1 Introduction

Control theory has a rich and lengthy history in engineering, while control theory for Markov models was first explored in the context of manpower planning in the works of [Bartholomew, 1967], [Davies, 1973] and [Vajda, 1975]. Later on, this research avenue was extended to non-homogeneous Markov chains [Vassiliou and Tsantas, 1984a], [Vassiliou and Tsantas, 1984b], [Vassiliou and Georgiou, 1990],[Georgiou and Vassiliou, 1992]. Furthermore, the concept of n-step maintainability was studied as well [Davies, 1975], [Guerry, 1991], [Haigh, 1992].

In a manpower Markov model the states correspond to homogeneous personnel groups for which it is assumed that all members have comparable transition probabilities. The proportional number of members in each of the states are gathered in a vector that is called the population structure at that particular time. Control theory deals with the question whether or not a specific population structure can be maintained by an appropriate choice of the parameters that are controllable. As to the means of control, the following three categories of flows are considered [Bartholomew, 1967]:

- Wastage, denoted by the vector $\mathbf{w} = (w_i)$ where w_i is the probability that a member of state i leaves the system within one time interval
- Internal transitions, i.e. promotion and demotion, denoted by the matrix $\mathbf{P}^I$ where $\mathbf{P}^I_{ij}$ is the probability of the internal transition from state i to state j within one time interval
- Recruitment, denoted by the vector $\mathbf{r} = (r_i)$ where r_i is the probability that a new members enters the system in state i

Although those three categories can be used in order to exert control over the population structure, control by recruitment is often seen as the most attractive means of control. Using wastage as a means of control by dismissing people is frequently seen as unethical and can lead to a decrease in moral and job satisfaction. Whereas adjusting the promotion and demotion rates within an organization might create dissatisfaction among those who feel their anticipated career progression is being undermined. Furthermore, it may involve the promotion of inadequately qualified people or the demotion of the most proficient employees. Hence, control through recruitment is often favored because it can be implemented without immediate repercussions for those currently employed Bartholomew [1967]. However, minor adaptations concerning the means of control, such as the use of the concept of pressure in the grades, which was introduced in [Kalamatianou, 1987] or the use of restricted recruitment as in [Ossai, 2013], were considered throughout the literature. Additionally, [Georgiou, 1992] introduced the notion of partial maintainability, which entails the need to retain a portion of the population structure while allowing for flexibility and variability in another portion. Control theory for semi-Markov processes did not get much attention in the literature. Maintainability of the state sizes in the case of a non-homogeneous semi-Markov process was first studied in [Vassiliou and Papadopoulou, 1992], where Vassiliou and Papadopoulou extended the concept of maintainability by imposing that the number of members is maintained for each seniority class within a grade.

We will concentrate our attention to the case of discrete-time homogeneous semi-Markov chains, building on the ideas of [Davies, 1973] and [Vassiliou and Papadopoulou, 1992]. We introduce a more comprehensive definition of maintainability for semi-Markov models, i.e. the State Reunion Maintainability with respect to a matrix $\mathbf{M}$. Afterwards, we uncover the inherent connection between the State Reunion Maintainability of discrete time homogeneous semi-Markov chains and the (classical) maintainability for non-homogeneous Markov chains.

It should be noted that although we follow the historical trend to introduce control theory in Markov processes by using the terminology of manpower planning, one can use this framework in other (semi-) Markov systems consisting of incoming flows, internal transitions and outgoing flows to an absorbing state.

In the remainder of this text, we direct our attention to control by recruitment in line with previous work of [Davies, 1973], [Vassiliou and Tsantas, 1984a] As a consequence, we consider the wastage as well as the internal transition probabilities as fixed and assume that the recruitment vector is under control of management.

The following section provides the essential background information concerning maintainability under control by recruitment for a Markov chain.

2 Maintainability under control by recruitment for a Markov chain

Let us consider a Markov system with l organizational states $S_1, ..., S_l$. The stock vector of the system at time t is the vector with i-th element the number of members in S_i at time t. By expressing the stocks proportionally, one obtains a probability vector $\mathbf{s}(t)$ that is called the population structure of the system at time t. Denoting the internal transition matrix as $\mathbf{P}_M^I$, the wastage vector as $\mathbf{w}$ and the recruitment vector, which is a probability vector, as $\mathbf{r}$, for a system with constant total size, the evolution of the population structure can be described as in [Bartholomew, 1967]:

$$\mathbf{s}(t+1) = \mathbf{s}(t)(\mathbf{P}_M^I + \mathbf{w}'\mathbf{r})$$

and a structure $\mathbf{s}$ is maintainable under control by recruitment if and only if

$$\mathbf{s} \geq \mathbf{s}\,\mathbf{P}_M^I$$

It is important to emphasize that the internal transition matrix $\mathbf{P}_M^I$ as we consider here, is a substochastic matrix. It becomes a probability matrix by extending the set of organizational states, which are homogeneous personnel groups within the system, with one or more additional absorbing state(s): the wastage state(s). This extended stochastic matrix is denoted by $\mathbf{P}_M^S$.

Remark 2.1. For a maintainable structure $\mathbf{s}$ we can define the non-normalized recruitment vector $\mathbf{r}^*$ as follows:

$$\mathbf{r}^* := \mathbf{s} - \mathbf{s}\,\mathbf{P}_M^I$$

By normalizing the vector $\mathbf{r}^*$ with respect to the L^1 norm, one gets the recruitment vector $\mathbf{r}$.

Furthermore, in [Bartholomew, 1967] the set of maintainable structures is fully described as the convex hull of its vertices:

Theorem 2.1. *[Bartholomew, 1967] Suppose that $\mathbf{P}_M^I$ is the internal transition matrix of a Markov system. The maintainable region under control by recruitment is the convex set with vertices given by the normalised rows of $(\mathbf{I}-\mathbf{P}_M^I)^{-1}$, with respect to the L^1 norm.*

The matrix $\mathbf{I}$ is the $l \times l$ identity matrix. Since $\mathbf{P}_M^I$ is a substochastic matrix the series $\mathbf{I} + \mathbf{P}_M^I + \left(\mathbf{P}_M^I\right)^2 + ...$ converges to $(\mathbf{I} - \mathbf{P}_M^I)^{-1}$. Which not only ensures that all entries of $(\mathbf{I} - \mathbf{P}_M^I)^{-1}$ are non-negative but guarantees the invertibility of $\mathbf{I} - \mathbf{P}_M^I$. Hence, each row of $(\mathbf{I} - \mathbf{P}_M^I)^{-1}$ is a non-negative vector that, after normalization by using the L^1 norm, results in a probability vector. Those vectors span the set of maintainable structures.

3 Maintainability under control by recruitment for a semi-Markov chain

When using semi-Markov chains, the main goal is often to estimate the semi-Markov kernel $\mathbf{q}$ [Barbu and Limnios, 2009]. Suppose that the organisational states of a semi-Markov chain are given by $\mathbf{S} = \{S_1, S_2, \cdots, S_l\}$ and let T_n and J_n denote the time of the n-th transition and the state occupied after the n-th transition, respectively. The semi-Markov kernel $\mathbf{q} = (q_{ij}(k) : i, j \in \mathbf{S}, k \in \mathbb{N})$ where

$$q_{ij}(k) = \Pr(J_{n+1} = S_j, T_{n+1} - T_n = k | J_n = S_i).$$

This semi-Markov kernel can be used to construct a sequence of matrices $\{\mathbf{P}(k)\}_{0 \le k \le K}$ where $\mathbf{P}(k)$ corresponds to the the one-step ahead transition matrix for staff with grade seniority $k \in \{0, ..., K\}$, where K denotes the maximal attainable state seniority, i.e. the maximal duration of stay in an organisational state.

Theorem 3.1. *[Verbeken and Guerry, 2021] For all k such that $\sum_{h\in\mathbf{S}} \sum_{l=0}^{k} q_{ih}(l) \neq 1$ we have:*

$$\mathbf{P}_{ij}(k) = \begin{cases} \dfrac{q_{ij}(k)}{1 - \sum_{h\in\mathbf{S}} \sum_{l=0}^{k} q_{ih}(l)} & \text{if } i \neq j \\ 1 - \sum_{i\neq j} \dfrac{q_{ij}(k)}{1 - \sum_{h\in\mathbf{S}} \sum_{l=0}^{k} q_{ih}(l)} & \text{if } i = j \end{cases}$$

Remark 3.1. Following the classical notation [Barbu and Limnios, 2009], writing $N_{ij}(k)$ for the number of persons in state S_i with grade seniority k that go to state S_j during the next time step, while writing $N_i(k)$ for the total number of people in state S_i with grade seniority k, the following estimator can be used for $\mathbf{P}_{ij}(k)$:

$$\widehat{\mathbf{P}_{ij}(k)} = \frac{N_{ij}(k)}{N_i(k)} \tag{1}$$

First of all, in order to study the maintainability of a semi-Markov chain, we need to combine all of the information in the matrices $\mathbf{P}(k)$ into one matrix $\mathbf{P}_{SM}$ that characterizes the semi-Markov (SM) model. This can be done by breaking up the states $\mathbf{S}$ according to grade seniority. The matrix $\mathbf{P}_{SM}$ is then the transition matrix in accordance with these seniority based disaggregated states.

Definition 3.1. Suppose we have l organisational states and one or more states that correspond to leaving the system (absorbing states often called wastage states). If the sequence $\{\mathbf{P}(k)\}_k$ is of length $K+1$, then $\mathbf{P}_{SM}$ is the matrix with:

$$(\mathbf{P}_{SM})_{ij} = 0 \quad \text{for } k \neq i-1 \mod (K+1) \tag{2}$$

and, if $k = i-1 \mod (K+1)$:

$$(\mathbf{P}_{SM})_{ij} = \begin{cases} \mathbf{P}(k)_{\lceil \frac{i}{K+1} \rceil, \lceil \frac{i}{K+1} \rceil} & \text{if } \lceil \frac{i}{K+1} \rceil = \lceil \frac{j}{K+1} \rceil \quad \text{and } (j-1-i) \equiv_{K+1} 0 \\ \mathbf{P}(k)_{\lceil \frac{i}{K+1} \rceil, \lceil \frac{j}{K+1} \rceil} & \text{if } \lceil \frac{i}{K+1} \rceil \neq \lceil \frac{j}{K+1} \rceil \quad \text{and } (j-1) \equiv_{K+1} 0 \end{cases} \tag{3}$$

In general, if we describe a transition from i to j, $(i-1) \mod (K+1)$ corresponds to the grade seniority k in the organisational state where the transition starts from, while $\lceil \frac{i}{K+1} \rceil$ and $\lceil \frac{j}{K+1} \rceil$ correspond to the organisational states themselves. This implies that equation (3) expresses the two possible types of internal transitions: Either, the organisational state is preserved, which consequently means that the grade seniority increases by one, or the organisational state changes, which implies that the grade seniority is reset to zero.

Example 3.1. Suppose that we have a sequence of 4×4 one-step ahead transition matrices $\{\mathbf{P}(k)\}_{0 \leq k \leq 3}$ of length 4 with 3 internal states and one wastage state, so $l = 3$. Then according to (3) it follows that

$$\mathbf{P}_{SM} = \begin{bmatrix} 0 & P_{11}(0) & 0 & 0 & P_{12}(0) & 0 & 0 & 0 \\ 0 & 0 & P_{11}(1) & 0 & P_{12}(1) & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{11}(2) & P_{12}(2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{12}(3) & 0 & 0 & 0 \\ P_{21}(0) & 0 & 0 & 0 & 0 & P_{22}(0) & 0 & 0 \\ P_{21}(1) & 0 & 0 & 0 & 0 & 0 & P_{22}(1) & 0 \\ P_{21}(2) & 0 & 0 & 0 & 0 & 0 & 0 & P_{22}(2) \\ P_{21}(3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

Redefining the state set may be more practical in this context. Therefore, we define the following:

Definition 3.2. The seniority-based states are given by

$$\mathbf{S_{SB}} = \{S_{a(b)} \mid 0 \leq a \leq K \text{ and } 1 \leq b \leq l\},$$

where the state $S_{a(b)}$ corresponds to the staff with grade seniority a in organisational state b.

This allows us to consider $\mathbf{P}_{SM}$ as the transition matrix with state space $\mathbf{S_{SB}}$. As is the case for the Markov system, $\mathbf{P}_{SM}$ is no probability matrix but can easily be made into one by supplementing the set of seniority-based states with the absorbing wastage states and the corresponding probabilities.

If we write $\mathbf{s_{SB}}$ for the vector with entries the proportion of the stocks in the seniority-based states, $\mathbf{w}$ for the wastage vector, and $\mathbf{r}$ for the recruitment vector, we obtain the following equation for a system with constant total size:

$$\mathbf{s_{SB}} = \mathbf{s_{SB}}\mathbf{P}_{SM} + \mathbf{s_{SB}}\mathbf{w}^{'}\mathbf{r} \tag{5}$$

that is equivalent with:

$$\mathbf{s_{SB}} = \mathbf{s_{SB}}\mathbf{w}^{'}\mathbf{r}(\mathbf{I} - \mathbf{P}_{SM})^{-1} \tag{6}$$

Now the question is, what should be maintained? Which proportions should be preserved? In the case of a Markov chain, often, a fixed proportion for each of the states is to be preserved. We could do the same in the semi-Markov case, but this would mean that the proportions of all seniority-based states should be preserved. This is in line with the approach in Vassiliou and Papadopoulou [1992], where this notion of maintainability was introduced for non-homogeneous semi-Markov chains. It is important to observe that this particular approach can be interpreted as an edge case of the concept of maintainability, which involves the preservation of proportions across all seniority-based states characterized by varying levels of seniority. In the context of real-world applications, it might often be deemed more prudent and efficacious to focus on preserving select combinations of states. This leads to a generalised and more comprehensive definition of maintainability:

Definition 3.3. (Fusion Matrix)
A matrix $\mathbf{M} = (m_{ij})$ is called a fusion matrix, if and only if $m_{ij} \in \{0, 1\}$ for all i, j and if every row of $\mathbf{M}$ contains exactly one element equal to 1.

Definition 3.4. (M-State Reunion Maintainability)
A semi-Markov structure $\mathbf{s}$ is called M-State Reunion Maintainable if the following equation holds for at least one probability vector $\mathbf{r}$

$$\mathbf{s} =: \mathbf{s_{SB}}\mathbf{M} = \mathbf{s_{SB}}\mathbf{w}^{'}\mathbf{r}(\mathbf{I} - \mathbf{P}_{SM})^{-1}\mathbf{M} \tag{7}$$

In a context without ambiguity, it is deemed appropriate to employ the abbreviated form SR_M-maintainability for the sake of brevity and notational convenience.

Remark that recruitment is only allowed in seniority-based states with zero seniority. So the recruitment vector $\mathbf{r}$ solely possesses non-zero entries for the seniority-based states with zero seniority.

The matrix $\mathbf{M}$ encodes which seniority-based states should fuse into states for which the population structure has to be maintained. As we only allow each seniority-based state to contribute to exactly one fused state, it follows that each row of $\mathbf{M}$ has to contain exactly one element equal to 1. For example, if we take $\mathbf{M} = \mathbf{I}$, we want to maintain the proportions of all seniority-based states. The other (but useless) extreme case, is the case where $\mathbf{M}$ is just the all-one vector. This means that all grade seniority-based states should fuse into one state and that the proportion of people in this state should be maintained. As we are working with proportions, this is quite useless as this will be always the case. Besides those two extreme cases, in an intermediate situation $\mathbf{M}$ has a format $(l \times (K+1), z)$ and on the j-th column $\left[\mathbf{M}\right]_j$ a 1 on the i-th place if and only if the i-th seniority-based-state is fused in the j-th reunited state.

We will be primarily interested in the case where we want to preserve the proportions of people in the organisational states. This means that, for each $b \in \mathbf{S}$ we fuse the states $\{S_{a(b)}\}_{0 \leq a \leq K}$ into one state. Which implies that $\mathbf{M}$ will be the matrix consisting of l columns, where each column $\left[\mathbf{M}\right]_j$ consists of $K+1$ ones through:

$$M_{ij} = \begin{cases} 1 & \text{if } (j-1)(K+1) \leq i \leq j(K+1) \\ 0 & \text{else} \end{cases} \tag{8}$$

4 A comparison between Markov and semi-Markov chains

A semi-Markov chain inherently comprises more information than a Markov chain. Suppose that we have a data set at our disposal which admits the estimation of a semi-Markov system. Then we can create the sequence of matrices $\{\mathbf{P}(k)\}_{0 \leq k \leq K}$ according to Theorem 3.1 and we can choose an appropriate matrix $\mathbf{M}$ to study the SR_M-maintainability of this system. Now suppose that we use the same data to estimate a Markov system and study the maintainability of that system.

Remark that it is possible to recreate the estimated Markov matrix starting from the $\{\mathbf{P}(k)\}_{0 \leq k \leq K}$ by using the proportions of the seniority-based states to the organisation states. This property is formulated in the following proposition:

Proposition 4.1. *Suppose that we have a sequence of $n \times n$ transition matrices $\{\mathbf{P}(k)\}_{0 \leq k \leq K}$ as in Theorem 3.1 which were estimated on real world data, where the states correspond to the seniority-based states $\mathbf{S_{SB}}$. If we estimate a Markov matrix $\mathbf{P}_M^S$ on the same data, where we use the organisational states $\mathbf{S}$ supplemented with one or more absorbing states, which we will call wastage states, we obtain that:*

$$\mathbf{P}_M^S = \sum_{k=0}^{K} \operatorname{diag}(\alpha_k^1, \alpha_k^2, \ldots, \alpha_k^n) \cdot \mathbf{P}(k) \tag{9}$$

Where $\operatorname{diag}(\alpha_k^1, \alpha_k^2, \ldots, \alpha_k^n)$ *corresponds to an $n \times n$ diagonal matrix with diagonal elements $\alpha_k^1, \alpha_k^2, \ldots, \alpha_k^n$, and where α_k^i equals the proportion of the number of people in state S_i with grade seniority k to the total number of people in the state S_i in the data.*

Proof. Following the classical notation [Barbu and Limnios, 2009], writing N_{ij} for the total number of people which make the transition from S_i to S_j in one time-step and writing N_i for the total number of people in state S_i, we infer that

$$\alpha_k^i = \frac{N_i(k)}{N_i} \tag{10}$$

The notations $N_{ij}(k)$ and $N_i(k)$ are as in Equation (1). Now we see that:

$$\sum_{k=0}^{K} \alpha_k^i \widehat{P_{ij}(k)} = \sum_{k=0}^{K} \alpha_k^i \frac{N_{ij}(k)}{N_i(k)} = \sum_{k=0}^{K} \frac{N_i(k)}{N_i} \frac{N_{ij}(k)}{N_i(k)} = \frac{1}{N_i} \sum_{k=0}^{K} N_{ij}(k) = \frac{N_{ij}}{N_i} = \left(\widehat{\mathbf{P}_M^S}\right)_{ij} \tag{11}$$

and Equation (9) follows. □

Remark 4.1. Remark that we used matrix notation to express the fact that the ith row of $\mathbf{P}_M^S$ can be seen as a convex combination of the ith rows of the matrices $\mathbf{P}(k)$.

5 State Reunion maintainability for semi-Markov chains and maintainability of non-homogeneous Markov chains

In this section, we establish the connection between SR_M maintainable structures for homogeneous semi-Markov chains and maintainable structures for non-homogeneous Markov chains by constructing a non-homogeneous Markov chain, $\{{}_\alpha\mathbf{P}_M^S(t)\}_{0\leq t\leq T}$ such that the structure $\mathbf{s}^*$ is SR_M maintainable for T-steps for the semi-Markov chain if and only if the structure $\mathbf{s}^*$ is maintainable for T-steps for the non-homogeneous Markov chain $\{{}_\alpha\mathbf{P}_M^S(t)\}_{0\leq t\leq T}$.

Theorem 5.1. *Given a semi-Markov chain with sequence of transition matrices $\{\mathbf{P}(k)\}_{0\leq k\leq K}$. The vector $\mathbf{s}^* = (s_1, s_2, \cdots, s_n)$ is SR_M-maintainable for T-steps if and only if there exists, for every $t \in \{0, 1, 2, \cdots, T\}$ a list of weighting vectors $\alpha^m(t)_{1\leq m\leq n}$ such that the vector $\mathbf{s}^*$ is maintainable for T-steps for the non-homogeneous Markov chain $\{{}_\alpha\mathbf{P}_M^S(t)\}_{0\leq t\leq T}$, where*

$${}_\alpha\mathbf{P}_M^S(t) = \sum_{k=0}^{K} \mathrm{diag}(\alpha_k^1(t), \alpha_k^2(t), \cdots, \alpha_k^n(t)) \cdot \mathbf{P}(k)$$

Proof. We will prove this statement by inductively constructing the sequence of matrices corresponding to the non-homogeneous Markov chain $\{{}_\alpha\mathbf{P}_M^S(t)\}_{0\leq t\leq T}$. First, suppose that $\mathbf{s}^*$ is SR_M-maintainable. In particular, according to definition 3.4, it follows that there exists a vector $\mathbf{s}^*_{\mathbf{SB}}(0)$ such that

$$\mathbf{s}^* = \mathbf{s}^*_{\mathbf{SB}}(0)\mathbf{M} \tag{12}$$

We know that the matrix $\mathbf{M}$ encodes combinations of seniority-based states which are fused together. This implies that we can split $\mathbf{s}^*_{\mathbf{SB}}(0)$ into n parts, where, for each $m \in \{1, 2, \cdots, n\}$ the m-th part corresponds to the m-th state after fusioning. We denote this part as $\mathbf{s}^*_{\mathbf{SB}}(0)[m]$. With each of these parts $\mathbf{s}^*_{\mathbf{SB}}(0)[m]$ we can associate a vector of length $K+1$: $\alpha^m(0) = (\alpha^m_0(0), \ldots, \alpha^m_K(0))$ which we will call the row-weighting vector associated with state m at time $t = 0$ and which is the normalisation of $\mathbf{s}^*_{\mathbf{SB}}(0)[m]$ with respect to the the L^1-norm. In particular,

$$\sum_{k=0}^{K} \alpha^m_k(0) = 1 \tag{13}$$

Now define the matrix ${}_{\alpha}\mathbf{P}^S_M(0)$ as follows:

$${}_{\alpha}\mathbf{P}^S_M(0) := \sum_{k=0}^{K} \mathrm{diag}(\alpha^1_k(0), \alpha^2_k(0), \cdots, \alpha^n_k(0)) \cdot \mathbf{P}(k) \tag{14}$$

As each row of the matrix ${}_{\alpha}\mathbf{P}^S_M(0)$ is a convex combination of the rows of $\mathbf{P}(k)$, which are probability vectors, all rows of ${}_{\alpha}\mathbf{P}^S_M(0)$ are probability vectors themselves, which implies that ${}_{\alpha}\mathbf{P}^S_M(0)$ is a probability matrix. Furthermore, it follows by construction that:

$$\mathbf{s}^* \cdot {}_{\alpha}\mathbf{P}^I_M(0) = \mathbf{s}^*_{\mathbf{SB}}(0)\mathbf{P}_{SM}\mathbf{M} \tag{15}$$

where ${}_{\alpha}\mathbf{P}^I_M(0)$ is the submatrix of ${}_{\alpha}\mathbf{P}^S_M(0)$ as a result of deleting the rows and columns related to the absorbing wastage state(s). The right-hand side of this equation corresponds to going one-step ahead with the seniority-based states, followed by fusioning those states according to $\mathbf{M}$, while the left-hand side can be seen as fusioning the matrices themselves, followed by going one-step ahead.

Now, if either $\mathbf{s}^*$ is SR_M-maintainable or $\mathbf{s}^*$ is maintainable for ${}_{\alpha}\mathbf{P}^S_M(0)$ we can define an appropriate $\mathbf{r}^*(0)$ and $\mathbf{r}^*_{\mathbf{SB}}(0)$ with

$$\mathbf{r}^*(0) = \mathbf{r}^*_{\mathbf{SB}}(0)\mathbf{M} \tag{16}$$

such that

$$\mathbf{s}^*_{\mathbf{SB}}(1) := \mathbf{s}^*_{\mathbf{SB}}(0)\mathbf{P}_{SM} + \mathbf{r}^*_{\mathbf{SB}}(0) \tag{17}$$

where

$$\mathbf{s}^* = \mathbf{s}^*_{\mathbf{SB}}(1)\mathbf{M} \tag{18}$$

This implies that $\mathbf{s}^*$ is SR_M-maintainable for one step if and only if $\mathbf{s}^*$ is maintainable for ${}_{\alpha}\mathbf{P}^S_M(0)$.

As long as $t \leq T$, we can repeat this procedure and inductively define $\alpha^m(t)$, ${}_{\alpha}\mathbf{P}^S_M(t), \mathbf{r}^*_{\mathbf{SB}}(t)$ and $\mathbf{s}^*_{\mathbf{SB}}(t)$ which concludes the proof. □

Remark 5.1. Observe the similarity between Equations (9) and (14). In Equation (9) the coefficients are determined based on all data in the database, whereas Equation (14) considers the coefficients at the specific point in time $t = 0$.

Theorem 5.1 makes it able to convert the problem of SR_M maintainability for semi-Markov chains to the study of maintainability for non-homogeneous Markov chains, which can be tackled with the tools developed in Vassiliou and Tsantas [1984b].

5.1 Illustration

This section illustrates how a homogeneous semi-Markov chain can be converted into a non-homogeneous Markov chain according to Theorem 5.1.
Suppose that the semi-Markov process is given by the following $\{\mathbf{P}(k)\}_{0\leq k\leq 2}$:
$\mathbf{P}(0)=\begin{pmatrix}0.2&0.5&0&0.3\\0&0.7&0.2&0.1\\0&0&0.9&0.1\\0&0&0&1\end{pmatrix}$, $\mathbf{P}(1)=\begin{pmatrix}0.6&0.3&0&0.1\\0&0.5&0.45&0.05\\0&0&0.9&0.1\\0&0&0&1\end{pmatrix}$ and
$\mathbf{P}(2)=\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}$
we obtain, following definition 3.1:

$$\mathbf{P}_{SM}=$$

	$S_{0(1)}$	$S_{1(1)}$	$S_{2(1)}$	$S_{0(2)}$	$S_{1(2)}$	$S_{2(2)}$	$S_{0(3)}$	$S_{1(3)}$	$S_{2(3)}$
$S_{0(1)}$	0	0.2	0	0.5	0	0	0	0	0
$S_{1(1)}$	0	0	0.6	0.3	0	0	0	0	0
$S_{2(1)}$	0	0	0	0	0	0	0	0	0
$S_{0(2)}$	0	0	0	0	0.7	0	0.2	0	0
$S_{1(2)}$	0	0	0	0	0	0.5	0.45	0	0
$S_{2(2)}$	0	0	0	0	0	0	0	0	0
$S_{0(3)}$	0	0	0	0	0	0	0	0.9	0
$S_{1(3)}$	0	0	0	0	0	0	0	0	0.9
$S_{2(3)}$	0	0	0	0	0	0	0	0	0

and $\mathbf{w}=(0.3,0.1,1,0.1,0.05,1,0.1,0.1,1)$.

Starting from $\mathbf{s}^*_{\mathbf{SB}}(0)=(0.1,0.1,0,0.2,0.1,0.1,0.1,0.1,0.2)$, with $\mathbf{M}=\begin{pmatrix}1&0&0\\1&0&0\\1&0&0\\0&1&0\\0&1&0\\0&1&0\\0&0&1\\0&0&1\\0&0&1\end{pmatrix}$
we infer that $\mathbf{s}^*=(0.2,0.4,0.4)$ with $\alpha^1(0)=(\frac{1}{2},\frac{1}{2},0)$, $\alpha^2(0)=(\frac{1}{2},\frac{1}{4},\frac{1}{4})$ and $\alpha^3(0)=(\frac{1}{4},\frac{1}{4},\frac{1}{2})$. As

$$_\alpha\mathbf{P}^S_M(0):=\sum_{k=0}^{3}\mathrm{diag}(\alpha^1_k(0),\alpha^2_k(0),\alpha^3_k(0))\cdot\mathbf{P}(k)=\begin{pmatrix}0.4&0.4&0&0.2\\0&0.475&0.2125&0.3125\\0&0&0.45&0.55\\0&0&0&1\end{pmatrix}$$

where

$$\mathbf{s}^* \cdot {}_\alpha\mathbf{P}_M^I(0) = \mathbf{s}^*_{\mathbf{SB}}(0)\mathbf{P}_{SM}\mathbf{M} = (0.08, 0.27, 0.265)$$

This implies that our starting vector is 1-step maintainable as we can define the non-normalised recruitment vector:

$$\mathbf{r}^*_{\mathbf{SB}}(0) = (0.12, 0, 0, 0.13, 0, 0, 0.135, 0, 0)$$

and the (normalised) recruitment vector

$$\mathbf{r}_{\mathbf{SB}}(0) = (0.311688, 0, 0, 0.337662, 0, 0, 0.350649, 0, 0)$$

So we obtain:

$$\mathbf{s}^*_{\mathbf{SB}}(1) := (0.12, 0.02.0.06, 0.21, 0.14, 0.05, 0.22, 0.09, 0.09) = \mathbf{s}^*_{\mathbf{SB}}(0)\mathbf{P}_{SM} + \mathbf{r}^*_{\mathbf{SB}}(0)$$

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