

Characterizing and computing weight-equitable partitions of graphs

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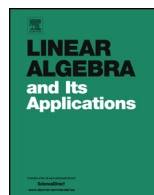


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Characterizing and computing weight-equitable partitions of graphs

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ABSTRACT

Weight-equitable partitions of graphs, which are a natural extension of the well-known equitable partitions, have been shown to be a powerful tool to weaken the regularity assumption in several classic eigenvalue bounds. In this work we aim to further our algebraic and computational understanding of weight-equitable partitions. We do so by showing several spectral properties and algebraic characterizations, and by providing a method to find coarse weight-equitable partitions.

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1. Introduction

Let $G = (V, E)$ be a simple undirected connected graph with n vertices, and let A denote its adjacency matrix. Many properties of G such as regularity or bipartiteness can be characterized from the spectrum of A . If G is large, however, investigating the

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spectrum of G might be cumbersome, which motivates to study “condensed” versions of A that preserve properties of its spectrum.

One of the most popular methods to shrink A is based on equitable partitions. To define this, let $\mathcal{P} = \{V_1, \dots, V_m\}$ ($m \leq n$) be a partition of V , and, for $i, j \in [m] := \{1, \dots, m\}$ and $u \in V_i$, let $b_{ij}(u)$ be the number of neighbors of u in V_j . The partition \mathcal{P} is called *equitable* (or *regular*) if $b_{ij}(u)$ is independent from the concrete choice of $u \in V_i$, i.e., $b_{ij}(u) = b_{ij}(v)$ for all $u, v \in V_i$. In this case, the $m \times m$ matrix $B = (b_{ij})$ is called the *quotient matrix* of \mathcal{P} . Since it is known that for an equitable partition the eigenvalues of B are also eigenvalues of A , see Godsil and Royle [15], some spectral properties of B carry over to A . Equitable partitions have been proven to be useful to derive, among many others, sharp eigenvalue bounds on the independence number like the celebrated ratio bound by Hoffman [19], but such results only hold when the underlying graph is regular.

To be able to derive graph properties from the spectrum also in the broader context of general graphs, a natural generalization of equitable partitions is to assign each vertex of G a weight such that G is “weight regularized”, which leads to the concept of weight-equitable partitions. Weight-equitable partitions have been shown to be a powerful tool to extend several classical results for non-regular graphs. They were first used by Haemers in 1979 [17, Theorem 6] to provide a proof of Hoffman’s lower bound for the chromatic number of a general graph, weakening the regularity assumption required in the well-known Hoffman result on the independence number [19]. Such a bound for general graphs has recently been extended to the distance k -chromatic number using weight-equitable partitions, see [3, Theorem 4.3]. Fiol and Garriga [10,8] used them to obtain several sharp spectral bounds for parameters of non-regular graphs. Examples of such results are an extension of Hoffman’s ratio bound for the chromatic number or a generalization of the Lovász bound for the Shannon capacity of a graph. Moreover, Fiol [8] used weight-equitable partitions to show that a bound for the weight-independence number is best possible, and Fiol and Garriga [10] used them to obtain spectral characterizations of distance-regularity around a set and spectral characterizations of completely regular codes. Recently, new algebraic characterizations of weight-equitable partitions and a new application of such partitions to improve the classical Hoffman’s ratio bound were shown in [2].

Note that there is a trade-off between the two main goals of (weight-) equitable partitions. On the one hand, the coarser the partition \mathcal{P} is (i.e., the smaller m), the smaller the subset of the spectrum of A that can be recovered from the spectrum of B . On the other hand, the finer the partition \mathcal{P} is (i.e., the larger m), the more information on the spectrum of A that can be recovered from the spectrum of B . Depending on the aimed result, one might be interested in either of the two extremes. If one is mostly concerned about shrinking A as much as possible, one is interested in finding a coarsest (weight-) equitable partition. For instance, for bounding the independence number of a graph, one only needs a weight partition with $m = 2$ cells (see Fiol [8]), but for characterizing pseudo-distance-regular graphs one needs to weight partition the graph into $m = d + 1$

cells, where $d+1$ is the number of distinct eigenvalues (see Fiol [9]). Another application of the coarsest equitable partition in linear programming was shown by Grohe et al. [16]. While for the equitable case, Bastert [4] showed that a coarsest equitable partition can be found very efficiently, we are not aware of any result in this direction for weight-equitable partitions.

The aim of this article is thus to better understand weight-equitable partitions from the algebraic and computational point of view, and to develop means to find coarse weight-equitable partitions. To this end, we derive novel algebraic characterizations of weight-equitable partitions. Several known characterizations of equitable partitions follow as a corollary of our results. Moreover, we devise an operator that turns fine weight-equitable partitions into coarser ones. Since devising an algorithm to find a coarsest non-trivial weight-equitable partition is open, we investigate the potential of this operator in producing coarse partitions via computational experiments. Our computational results show that this operator is able to produce very coarse partitions in many cases, allowing to achieve a significant reduction of the size of A .

The outline of this article is as follows. Section 2 introduces our notation as well as basic definitions. In Section 3, we derive spectral properties of weight-equitable partitions, whereas Section 4 provides novel characterizations of weight-equitable partitions and operators to generate them. Section 5 investigates the potential of one such operator to produce coarse weight-equitable partitions.

2. Basic definitions and notation

Throughout this article, we denote by I an identity matrix, by \mathbf{J} an all-ones matrix and by $\mathbf{1}$ an all-ones vector whose dimensions will be clear from the context. The i -th canonical vector (of suitable dimension) is denoted by e_i , and $\|\cdot\|$ denotes the Euclidean norm of a vector. Moreover, for a finite set V , we denote its powerset by $\mathfrak{P}(V)$. For a simple undirected connected graph $G = (V, E)$, we denote by $n = |V|$ the number of its vertices and by $A = A(G)$ its adjacency matrix. Throughout this article, we always assume G to be simple, undirected, and connected if not stated differently. Moreover, we assume the vertices to be labeled $1, \dots, n$, i.e., $V = [n]$. The set $G(u)$ denotes the neighborhood of a vertex $u \in V$, i.e., the set of vertices adjacent to u , and we write $u \sim v$ if $u, v \in V$ are adjacent. The automorphism group of G is denoted by $\text{Aut}(G)$.

The eigenvalues of A are given by $\lambda_1, \dots, \lambda_n$, and we assume from now on that the eigenvalues are sorted non-increasingly, i.e., $\lambda_1 \geq \dots \geq \lambda_n$. We denote the *spectrum* of G by

$$\text{sp}(G) := \text{sp}(A) := \{\theta_0^{m_0}, \theta_1^{m_1}, \dots, \theta_d^{m_d}\},$$

where $\theta_0 > \theta_1 > \dots > \theta_d$ are the distinct eigenvalues of A in decreasing order with multiplicities $m_i = m(\theta_i)$, $i \in \{0\} \cup [d]$. Note that $\theta_0 = \lambda_1$ and $\theta_d = \lambda_n$. Since G is connected (so A is irreducible), the Perron-Frobenius Theorem assures that λ_1 is

simple, positive, and has a positive eigenvector. If G is disconnected, the existence of such an eigenvector is not guaranteed, unless all its connected components have the same maximum eigenvalue. Throughout this work, the positive eigenvector associated with the largest (positive and with multiplicity one) eigenvalue λ_1 is denoted by $\nu = (\nu_1, \dots, \nu_n)^\top$. This eigenvector is called the *Perron eigenvector*, and we assume it to be normalized such that its minimum entry is 1. For instance, if G is regular, we have $\nu = \mathbf{1}$.

To be able to define weight-equitable partitions of a connected simple graph G with Perron eigenvector ν , we consider the map $\rho: \mathfrak{P}(V) \rightarrow \mathbb{R}^n$, defined by $\rho(U) := \sum_{u \in U} \nu_u e_u$ for any $U \neq \emptyset$. By convention, $\rho(\emptyset) = 0$, and we write $\rho(u)$ instead of $\rho(\{u\})$. Since ρ is linear, we can interpret it to assign each $u \in V$ the weight $\rho(u) = \nu_u$. Doing so, we “regularize” the graph, in the sense that the *weight-degree* δ_u^* of each vertex $u \in V$ becomes a constant, where

$$\delta_u^* := \frac{1}{\nu_u} \sum_{v \in G(u)} \nu_v = \lambda_1.$$

If \mathcal{P} is a partition of the vertex set $V = V_1 \cup \dots \cup V_m$, the *weight-intersection number* of $u \in V_i$, $i \in [m]$, is

$$b_{ij}^*(u) := \frac{1}{\nu_u} \sum_{v \in G(u) \cap V_j} \nu_v, \quad i, j \in [m].$$

Observe that the sum of the weight-intersection numbers for all $j \in [m]$ gives the weight-degree of each vertex $u \in V_i$:

$$\sum_{j=1}^m b_{ij}^*(u) = \frac{1}{\nu_u} \sum_{v \in G(u)} \nu_v = \delta_u^* = \lambda_1.$$

Using these definitions, we are now able to define weight-equitable partitions.

Definition 1. Let G be a connected simple graph and let $\mathcal{P} = (V_1, \dots, V_m)$ be a partition of V . Then, \mathcal{P} is called *weight-equitable* (or *weight-regular*) if $b_{ij}^*(u) = b_{ij}^*(v)$ for all $i, j \in [m]$ and $u, v \in V_i$. That is, the weight-intersection numbers do not depend on the vertex $u \in V_i$. In this case, we write b_{ij}^* instead of $b_{ij}^*(u)$, $u \in V_i$.

A matrix characterization of weight-equitable partitions can be done via the following matrix associated with any partition \mathcal{P} . The *weight-characteristic matrix* of \mathcal{P} is the $n \times m$ matrix $\tilde{S}^* = (\tilde{s}_{uj}^*)$ with entries

$$\tilde{s}_{uj}^* = \begin{cases} \nu_u, & \text{if } u \in V_j, \\ 0, & \text{otherwise,} \end{cases}$$

Table 1

Some particular cases of trivial partitions. Note that for weight-equitable partitions the coarsest partition is always trivial, while the finest partition is trivial for regular graphs.

number of cells m	graph class admitting ... partition with m cells	
	equitable	weight-equitable
1	\iff regular	all
2	biregular	bipartite
n	all	\iff regular

for all $(u, j) \in V \times [m]$ and, hence, satisfying $(\tilde{S}^*)^\top \tilde{S}^* = D^2$, where D denotes the diagonal matrix $\text{diag}(\|\rho(V_1)\|, \dots, \|\rho(V_m)\|)$.

From such a weight-characteristic matrix we define the *weight-quotient matrix* of A , with respect to \mathcal{P} , as $\tilde{B}^* := (\tilde{S}^*)^\top A \tilde{S}^* = (\tilde{b}_{ij}^*)$. Notice that this matrix is symmetric and has entries

$$\tilde{b}_{ij}^* = \sum_{u,v \in V} \tilde{s}_{ui}^* a_{uv} \tilde{s}_{vj}^* = \sum_{u \in V_i, v \in V_j} a_{uv} \nu_u \nu_v = \sum_{uv \in E(V_i, V_j)} \nu_u \nu_v = \tilde{b}_{ji}^*,$$

where $E(V_i, V_j)$ stands for the set of edges with ends in V_i and V_j (when $V_i = V_j$ each edge counts twice).

In this article we will use the *normalized weight-characteristic matrix* of \mathcal{P} , which is the $n \times m$ matrix $\bar{S}^* = (\bar{s}_{uj}^*)$ with entries obtained by normalizing the columns of \tilde{S}^* , that is, $\bar{S}^* = \tilde{S}^* D^{-1}$. Thus,

$$\bar{s}_{uj}^* = \begin{cases} \frac{\nu_u}{\|\rho(V_j)\|}, & \text{if } u \in V_j, \\ 0, & \text{otherwise,} \end{cases}$$

and it holds that $(\bar{S}^*)^\top \bar{S}^* = I$. We define the *normalized weight-quotient matrix* of A with respect to \mathcal{P} , $\bar{B}^* = (\bar{b}_{ij}^*)$, to be the $m \times m$ matrix

$$\bar{B}^* = (\bar{S}^*)^\top A \bar{S}^* = D^{-1} (\tilde{S}^*)^\top A \tilde{S}^* D^{-1} = D^{-1} \tilde{B}^* D^{-1},$$

and hence $\bar{b}_{ij}^* = \frac{\tilde{b}_{ij}^*}{\|\rho(V_i)\| \|\rho(V_j)\|}$.

In Table 1, some trivial cases of (weight-) equitable partitions are summarized. Note that for $m = 2$, it does not hold that a partition into two sets is weight-equitable if and only if it is a bipartition of the graph; there may be many other weight-equitable partitions and a graph which admits one may not be bipartite. For example, the path graph P_4 on 4 vertices has two weight-equitable partitions: its bipartition and the partition which groups the two endpoints and internal vertices. However, a bipartition is always weight-equitable.

The following characterization of weight-equitable partitions by the first author [2] will be used to prove our main results.

Lemma 2 ([2]). *Let A be the adjacency matrix of a connected graph G , and let \mathcal{P} be a weight-equitable partition of the vertex set of G with normalized weight-characteristic matrix \bar{S}^* . Then, \mathcal{P} is weight-equitable if and only if A and $\bar{S}^*(\bar{S}^*)^\top$ commute.*

In [2], it is shown that weight-regular partitions can be used to improve the well-known Hoffman ratio bound on the chromatic number of a graph. A graph coloring which satisfies this bound with equality is referred to as a *Hoffman coloring*. We take the opportunity to correct the statement of Proposition 5.3 (ii) in [2], which should say that if a graph G has chromatic number $\chi(G)$ and a Hoffman coloring, then it holds that the multiplicity of the smallest eigenvalue λ_n is at least $\chi(G) - 1$ (and not only equal), and equality implies a unique Hoffman coloring.

3. Spectral properties of weight-equitable partitions

As mentioned previously, the aim of weight-equitable partitions is to condense the adjacency matrix of an undirected graph to make assessing its spectrum easier. This section is devoted to, on the one hand, deriving properties of the condensed adjacency matrix that are independent from the weight-equitable partition. That is, these results provide conditions that are necessary for a partition to be weight-equitable. On the other hand, we give new insights into the relation of weight-equitable and equitable partitions by providing a necessary criterion such that weight-equitable partitions are also equitable.

Theorem 3. *Let G be a connected graph with adjacency matrix A . Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of A and \mathcal{P} be a weight-equitable partition for A . Let \bar{B}^* be the normalized weight-quotient matrix of A with respect to \mathcal{P} , with eigenvalues $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$. Then, $\lambda_1 = \mu_1$.*

Proof. Since G is connected, A is an irreducible matrix, which means that \bar{B}^* is also irreducible. Let $y = (y_1, \dots, y_m)$ be a positive eigenvector to μ_1 and let \bar{S}^* denote the normalized weight-characteristic matrix of \mathcal{P} . Note that Lemma 2.2 (a) from [13] can easily be extended to weight-equitable partitions if we replace B and P by \bar{B}^* and \bar{S}^* . Then the vector $x = \bar{S}^*y$ is a positive vector such that $Ax = \mu_1x$, implying that μ_1 is an eigenvalue of A with eigenvector x . Perron-Frobenius Theorem implies that $\mu_1 = \lambda_1$, completing the proof. \square

As consequence of Theorem 3, we obtain the analogous result for equitable partitions of graphs [13, Corollary 2.3], which states that the quotient matrix of an equitable partition has the same spectral radius as the adjacency matrix. We should observe that the above result also holds for non-negative symmetric matrices, see [1, Theorem 2.1].

Because of Theorem 3 we know that the largest eigenvalue of A and \bar{B}^* are the same for any weight-equitable partition of V . Based on this result, we can devise a necessary

criterion for a partition to be weight-equitable that can be tested by evaluating a single matrix-vector multiplication.

Lemma 4. *Let G be a connected graph and $\{V_1, \dots, V_m\}$ be a weight-equitable partition of its vertex set. Let λ_1 be the largest eigenvalue of G . Then, the corresponding normalized weight-quotient matrix \bar{B}^* has eigenvector $x = (\|\rho(V_1)\|, \dots, \|\rho(V_m)\|)^\top$ with eigenvalue λ_1 .*

Proof. For $i \in [n]$, the entries of \bar{B}^*x equal

$$\begin{aligned}
 (\bar{B}^*x)_i &= \sum_{j=1}^m \bar{b}_{ij}^* \|\rho(V_j)\| = \sum_{j=1}^m \frac{\sum_{u \in V_i} \nu_u^2 b_{ij}^*(u)}{\|\rho(V_i)\| \|\rho(V_j)\|} \|\rho(V_j)\| \\
 &= \sum_{u \in V_i} \frac{\nu_u^2}{\|\rho(V_i)\|} \sum_{j=1}^m b_{ij}^*(u) = \lambda_1 \|\rho(V_i)\| = \lambda_1 x_i. \quad \square
 \end{aligned}$$

Remark 5. The relation between the spectrum of A and \bar{B}^* has also been explored in the context of eigenvalue interlacing. Fiol [8] showed that, given a vertex partition, if the eigenvalues of \bar{B}^* tightly interlace those of A , then the corresponding partition must be weight-equitable, extending Haemers’ results on interlacing [18]. Note that, however, the converse is generally not true: weight-equitability does not imply tight interlacing, as shown for example by the path P_4 with its vertices partitioned alternatingly into two cells.

One of our main goals is to investigate means to find coarse weight-equitable partitions. Since every equitable partition is also weight-equitable (which follows trivially from [1, Lemma 2.2]), we can use for example Bastert’s algorithm [4] to compute a lower bound on the coarseness of a weight-equitable partition. One might thus wonder whether the converse can also be true, i.e., whether the coarsest non-trivial weight-equitable partition is also equitable. In general, this is not the case, but we are able to provide a necessary criterion. Note that a similar statement was proved in [2]. However, here we provide an equivalent formulation which characterizes equitability of weight-equitable partitions. A proof is included for the sake of completeness.

Proposition 6. *Let G be a connected graph with adjacency matrix A and positive eigenvector ν , and consider a weight-equitable partition of the vertex set $\mathcal{P} = \{V_1, \dots, V_m\}$ with normalized weight-characteristic matrix \bar{S}^* . Then, $\nu = (\nu_1 \mathbf{1}^\top, \dots, \nu_m \mathbf{1}^\top)^\top$, with $\mathbf{1}$ ’s being all-one vectors of length $|V_i|$ for $i \in [m]$, if and only if \mathcal{P} is equitable.*

Proof. Let $V_i, V_j \in \mathcal{P}$ and let $u, v \in V_i$ be arbitrary. Since \mathcal{P} is weight-equitable, it must hold that

$$b_{ij}^*(u) = \frac{1}{\nu_u} \sum_{w \in G(u) \cap V_j} \nu_w = \frac{1}{\nu_v} \sum_{w \in G(v) \cap V_j} \nu_w = b_{ij}^*(v).$$

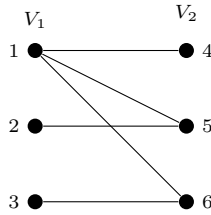


Fig. 1. A weight-equitable bipartition such that Perron eigenvector ν is not constant over each cell.

If ν is constant over every cell of \mathcal{P} , this implies that $|G(u) \cap V_j| = |G(v) \cap V_j|$, hence \mathcal{P} is equitable. Conversely, let \mathcal{P} be equitable with quotient matrix B and characteristic matrix S . It follows from [15, Lemma 9.3.1] that $SB = AS$. Then every eigenvector v of B gives an eigenvector $Sv = (v_1 \mathbf{1}^\top, \dots, v_m \mathbf{1}^\top)^\top$ of A , since $Bv = \lambda v$ implies that

$$A(Sv) = SBv = \lambda(Sv).$$

In particular, the Perron eigenvector of B gives the Perron eigenvector ν for A . \square

Note that, as a consequence of Proposition 6, all weight-equitable partitions of a regular graph are also equitable. For weight-equitable partitions which are not equitable, ν is not of the form requested in Proposition 6, as illustrated in Example 7.

Example 7. The bipartite graph $G = (V_1 \cup V_2, E)$ shown in Fig. 1 has Perron eigenvector $\nu \approx (2.732, 1, 1, 1.414, 1.932, 1.932)$, which is not constant for either cell V_i . However, it is easily checked that $\{V_1, V_2\}$ is a weight-equitable partition of G (see also Table 1).

Remark 8. A vertex partition of a graph G is called an *orbit partition* if its classes correspond to the orbits of (a subgroup of) $\text{Aut}(G)$. A graph is said to be *compact* if every doubly stochastic matrix which commutes with its adjacency matrix A is a convex combination of permutation matrices that commute with A . Godsil [14] showed that for compact graphs, all equitable partitions are orbit partitions. This result does not extend to weight-equitable partitions, as the trivial partition $\{V\}$ is always weight-equitable, see Table 1. Therefore, G must be vertex-transitive, and hence regular, if all weight-equitable partitions are also orbit partitions. In that case, ν is constant, so it follows from Proposition 6 that each weight-equitable partition is actually equitable.

4. Characterizations of weight-equitable partitions

In contrast to equitable partitions, no algorithmic procedure to find weight-equitable partitions has been discussed in the literature. To make progress in this direction, we derive novel characterizations of weight-equitable partitions and methods to generate weight-equitable partitions from known ones. We will investigate one of these methods from a practical point of view in Section 5.

Let $M \in \mathbb{R}^{m \times n}$. A pair of doubly stochastic matrices $(X, Y) \in \mathbb{R}^{m \times m} \times \mathbb{R}^{n \times n}$ is called a *fractional automorphism* of M if $XM = MY$. A fractional automorphism of type (X, X) is called a *fractional isomorphism*. Note that these definitions generalize the concept of graph automorphisms and isomorphisms from permutation matrices to doubly stochastic matrices.

In our first characterization, we link weight-equitable partitions of G to fractional isomorphisms of its adjacency matrix. Given a partition \mathcal{P} of $[n]$, we define $X_{\mathcal{P}} \in \mathbb{R}^{n \times n}$ to be the matrix with entries $x_{vv'} := \frac{\nu_v \nu_{v'}}{\|\rho(\mathcal{P})\|^2}$ if $v, v' \in P$ for some $P \in \mathcal{P}$ and $x_{vv'} := 0$ otherwise.

Proposition 9. *If \mathcal{P} is a weight-equitable partition of V , then $X_{\mathcal{P}}A = AX_{\mathcal{P}}$.*

Proof. This follows from Lemma 2, using the fact that $X_{\mathcal{P}} = \bar{S}^*(\bar{S}^*)^{\top}$. \square

Proposition 9 shows commutativity of a matrix derived from a weight-equitable partition. For equitable partitions, Godsil [14, Theorem 1.5] considered the converse. To this end, for a double stochastic matrix $X \in \mathbb{R}^{n \times n}$, define the directed graph G_X on n vertices with $n \times n$ adjacency matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } x_{ij} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The *strongly connected components* of a directed graph are the maximal induced subgraphs such that there exists a directed path between any two vertices. We define the *strongly connected components* of a doubly stochastic matrix X as the strongly connected components of the underlying graph G_X . Let \mathcal{P}_X denote the partition of $[n]$ into the strongly connected components of X . Since every equitable partition is also weight-equitable (with the exception of the partition into n cells, see Table 1), Godsil’s result also trivially applies to weight-equitable partitions:

Theorem 10. *If $X \neq I$ is a doubly stochastic matrix which commutes with $A(G)$, then the partition \mathcal{P}_X is weight-equitable.*

For equitable partitions, the matrix $X_{\mathcal{P}}$ is doubly stochastic, hence Theorem 9 and 10 imply that I is the only doubly stochastic matrix which commutes with $A(G)$ if and only if G has no nontrivial equitable partitions [14, Corollary 1.6]. In the case of weight-equitability, we cannot generalize Theorem 10 without extra assumptions, as the trivial partition is not necessarily weight-equitable, see Table 1.

In order to state our next result we need some preliminary definitions. Let $M \in \mathbb{R}^{V^0 \times W^0}$ and $N \in \mathbb{R}^{V^1 \times W^1}$ be two real-valued matrices. Two matrices M and N are said to be *fractionally isomorphic* if the following three properties hold:

- there are doubly stochastic matrices X, Y such that $X(M \oplus N) = (M \oplus N)Y$;

- for every $i \in \{0, 1\}$ and all $v \in V^i$, there exists $v' \in V^{1-i}$ with $x_{vv'} \neq 0$;
- for every $i \in \{0, 1\}$ and all $w \in W^i$, there exists $w' \in W^{1-i}$ with $y_{ww'} \neq 0$.

A *joint partition* of graphs G and H is a partition of $V(G) \cup V(H)$. A joint partition is *balanced* if every part has nonempty intersection with both $V(G)$ and $V(H)$.

The following result extends [16, Theorem 5.2] for weight-equitable partitions.

Theorem 11. *For all graphs G and H with adjacency matrices A and B , respectively, the following two statements are equivalent:*

- (i) *G and H have a balanced weight-equitable joint partition;*
- (ii) *the coarsest weight-equitable joint partition of G and H is balanced.*

Moreover, if $\rho: u \rightarrow \nu_u$ is constant over each cell of the partition, then (i) and (ii) imply that A and B are fractionally isomorphic.

Proof. Statement (i) follows immediately from (ii). Conversely, if there exists a balanced weight-equitable joint partition \mathcal{P} , it is a refinement of the coarsest one, hence any part P of the coarsest weight-equitable joint partition is the union of some $P_1, \dots, P_s \in \mathcal{P}$. Since each part P_i has nonempty intersection with $V(G)$ and $V(H)$, so does P .

To show that the above imply that A and B are fractionally isomorphic, let $\mathcal{P} = \{P_1, \dots, P_s\}$ be a balanced weight-equitable joint partition of G and H . By Lemma 2, the matrix $\bar{S}^*(\bar{S}^*)^\top$ satisfies $\bar{S}^*(\bar{S}^*)^\top(A \oplus B) = (A \oplus B)\bar{S}^*(\bar{S}^*)^\top$. Set $X = Y = \bar{S}^*(\bar{S}^*)^\top$ and let $P_i \in \mathcal{P}$, $v \in P_i$. Since we are assuming that $\rho: u \rightarrow \nu_u$ is constant over each cell of the partition, it holds that $X\mathbf{1} = \mathbf{1}X = \mathbf{1}$. Assume without loss of generality that $v \in V(G)$. Since \mathcal{P} is balanced, there exists a vertex $v' \in P_i$ such that $v' \in V(H)$. This means that $(\bar{S}^*(\bar{S}^*)^\top)_{vv'} \neq 0$. Therefore (X, Y) is a fractional isomorphism from A to B . \square

Observe that in order to link weight-equitable partitions to fractional isomorphism as we do in the last part of Theorem 11, one cannot avoid the assumption that $\rho: u \rightarrow \nu_u$ is constant over each cell of the partition, and actually this case is just equivalent to the equitable partition characterization that appeared in [16, Theorem 5.2]. Note that this is closely related with the result of Proposition 6. This is due to the fact that if a weight-equitable partition is equitable, then we know that $XA = AX, X\mathbf{1} = \mathbf{1}X = \mathbf{1}, X \geq 0$, see [14, Corollary 1.2]. If a weight-equitable partition is not equitable, then one can only guarantee $AX = XA, X\mathbf{1} = \mathbf{1}X, X \geq 0$ [2]. To the best of our knowledge, it remains an open problem to investigate the convex polytope that consists of all matrices X such that $XA = AX, X\mathbf{1} = \mathbf{1}X, X \geq 0$.

Let \mathcal{P} be a joint partition of graphs G and H . The *restriction* of \mathcal{P} to G is defined as $\mathcal{P}_G := \{P \cap V(G) \mid P \in \mathcal{P}\}$. We denote the intersection of a cell $P \in \mathcal{P}$ with $V(G)$ and $V(H)$ by P_G and P_H respectively.

Proposition 12. *Let G and H be graphs with coarsest weight-equitable joint partition \mathcal{P} . Then the restrictions of \mathcal{P} to G and H are the coarsest weight-equitable partitions of G and H .*

Proof. Without loss of generality, consider the restriction \mathcal{P}_G . Let $P, Q \in \mathcal{P}$ and fix an arbitrary vertex $u \in P_G$, then

$$b_{P,Q}^*(u) = \sum_{\substack{v \sim u \\ v \in Q}} \frac{\nu_v}{\nu_u} = \sum_{\substack{v \sim u \\ v \in Q_G}} \frac{\nu_v}{\nu_u} = b_{P_G, Q_G}^*(u).$$

Since \mathcal{P} is a weight-equitable partition, $b_{P,Q}^*(u)$ does not depend on u , hence \mathcal{P}_G is weight-equitable.

For the sake of contradiction, suppose that G has a coarser weight-equitable partition \mathcal{Q} . Let \mathcal{Q}' be the partition of $V(G) \cup V(H)$ which is given by

$$\mathcal{Q}' = \left\{ \bigcup_{\substack{P \in \mathcal{P} \\ P_G \subseteq Q}} P \mid Q \in \mathcal{Q} \right\}.$$

and consider two arbitrary cells $P, Q \in \mathcal{Q}'$. For a vertex $u \in P_G$, the weight-intersection number is given by $b_{P,Q}^*(u) = b_{P_G, Q_G}^*(u)$, which is independent of u , since \mathcal{Q} is weight-equitable. If $u \in P_H$, let $S \subseteq P$ denote the cell of \mathcal{P} containing u . Then

$$b_{P,Q}^*(u) = b_{P_H, Q_H}^*(u) = \sum_{\substack{T \subseteq Q \\ T \in \mathcal{P}}} b_{S_H, T_H}^*(u).$$

Note that for any $x \in S_G$, we have $b_{S_H, T_H}^*(u) = b_{S_G, T_G}^*(x)$, because \mathcal{P} is weight-equitable. It follows that

$$b_{P,Q}^*(u) = \sum_{\substack{T \subseteq Q \\ T \in \mathcal{P}}} b_{S_G, T_G}^*(x) = b_{S_G, Q_G}^*(x).$$

By weight-equitability of \mathcal{Q} , this number again equals $b_{P_G, Q_G}^*(x)$ and is independent of our choice of x and u . Therefore, \mathcal{Q}' is a coarser weight-equitable partition, a contradiction. \square

Proposition 13. *Let G and H be graphs with adjacency matrices A and B and balanced weight-equitable joint partition \mathcal{P} . If G is connected, then for all $P \in \mathcal{P}$,*

$$\frac{\|\rho(P_G)\|^2}{\|\rho(P_H)\|^2} = \frac{\|\rho(V(G))\|^2}{\|\rho(V(H))\|^2}.$$

Proof. Fix $P \in \mathcal{P}$ arbitrarily. Since G is connected, there exists a cell $Q \in \mathcal{P}$ such that $b_{P,Q}^* \neq 0$. We have shown in Proposition 12 that the restriction \mathcal{P}_G is again a weight-equitable partition with the same weight-intersection numbers. As a result,

$$b_{P,Q}^* \|\rho(P_G)\|^2 = b_{P_G, Q_G}^* \|\rho(P_G)\|^2 = \tilde{b}_{P,Q}^* = \tilde{b}_{Q,P}^* = b_{Q_G, P_G}^* \|\rho(Q_G)\|^2 = b_{Q,P}^* \|\rho(Q_G)\|^2.$$

Similarly, we can derive $b_{P,Q}^* \|\rho(P_H)\|^2 = b_{Q,P}^* \|\rho(Q_H)\|^2$. Combining both statements gives

$$\frac{b_{Q,P}^*}{b_{P,Q}^*} = \frac{\|\rho(P_G)\|^2}{\|\rho(Q_G)\|^2} = \frac{\|\rho(P_H)\|^2}{\|\rho(Q_H)\|^2},$$

which rewrites to

$$\frac{\|\rho(P_G)\|^2}{\|\rho(P_H)\|^2} = \frac{\|\rho(Q_G)\|^2}{\|\rho(Q_H)\|^2}. \tag{1}$$

Consider the graph G' on vertex set \mathcal{P} with edges $\{PQ \mid P, Q \in \mathcal{P}, b_{P,Q}^* \neq 0\}$. This is a connected graph, since G is connected. Note that the relation in Equation (1) is transitive: if $R \in \mathcal{P}$ such that $b_{Q,R}^* \neq 0$, then we have

$$\frac{\|\rho(P_G)\|^2}{\|\rho(P_H)\|^2} = \frac{\|\rho(Q_G)\|^2}{\|\rho(Q_H)\|^2} = \frac{\|\rho(R_G)\|^2}{\|\rho(R_H)\|^2}.$$

Since G' is connected, this means that Equation (1) holds for any $P, Q \in \mathcal{P}$. It follows that

$$\sum_{Q \in \mathcal{P}} \|\rho(Q_G)\|^2 \|\rho(P_H)\|^2 = \sum_{Q \in \mathcal{P}} \|\rho(Q_H)\|^2 \|\rho(P_G)\|^2,$$

hence

$$\frac{\|\rho(P_G)\|^2}{\|\rho(P_H)\|^2} = \frac{\|\rho(V(G))\|^2}{\|\rho(V(H))\|^2}. \quad \square$$

If one restricts to equitable partitions, then Propositions 12 and 13 give [16, Lemma 5.4] and [16, Lemma 5.6], respectively.

For stating the next characterization, we first need to introduce a linear operator. Let G be a connected graph with Perron eigenvector ν . Let \mathbb{V} be the vector space of all real functions on V , and, for any partition \mathcal{P} of V , let $F(V, \mathcal{P})$ be the subspace of \mathbb{V} that consists of all functions on V that are constant on the cells of \mathcal{P} . Consider the linear operator $\mathcal{B}: \mathbb{V} \rightarrow \mathbb{V}$ defined by

$$(\mathcal{B}f)(u) := \sum_{v \sim u} \frac{\nu_v}{\nu_u} f(u).$$

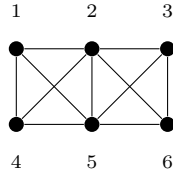


Fig. 2. A graph with two weight-equitable partitions, $\{\{1, 3, 5\}, \{2, 4, 6\}\}$ and $\{\{1, 5, 6\}, \{2, 3, 4\}\}$, whose meet is not weight-equitable.

Lemma 14. Let G be a connected graph and let $\mathcal{P} = \{V_1, \dots, V_m\}$ be a partition of V . Then, \mathcal{P} is weight-equitable if and only if $F(V, \mathcal{P})$ is \mathcal{B} -invariant.

Proof. The space $F(V, \mathcal{P})$ is spanned by the characteristic functions x_1, \dots, x_m which are one on cell $1, \dots, m$ respectively and zero elsewhere. It therefore suffices to show \mathcal{B} -invariance of this basis. Let $i, j \in [m]$ and $u \in V_i$. Then

$$(\mathcal{B}x_j)(u) = \sum_{v \sim u} \frac{\nu_v}{\nu_u} x_j(v) = \frac{1}{\nu_u} \sum_{v \in V_j \cap G(u)} \nu_v = b_{ij}^*(u).$$

This means that $\mathcal{B}x_j$ is constant on V_i if and only if $b_{ij}^*(u)$ is independent of u . Hence $\mathcal{B}x_j \in F(V, \mathcal{P})$ if and only if \mathcal{P} is weight-equitable. \square

The set of partitions of V can be viewed as a lattice where $\mathcal{Q} \leq \mathcal{P}$ if \mathcal{Q} is a refinement of \mathcal{P} . The meet $\mathcal{P} \wedge \mathcal{Q}$ of two partitions \mathcal{P}, \mathcal{Q} is the partition whose cells are the nonempty pairwise intersections of cells of \mathcal{P} and \mathcal{Q} . The join $\mathcal{P} \vee \mathcal{Q}$ is given by the connected components of the graph with edge set $\{\{u, v\} : u, v \text{ share a cell in } \mathcal{P} \text{ or } \mathcal{Q}\}$.

Corollary 15. Let \mathcal{P} and \mathcal{Q} be weight-equitable partitions of a connected graph G . Then the join $\mathcal{P} \vee \mathcal{Q}$ is a weight-equitable partition.

Proof. Since \mathcal{P} and \mathcal{Q} are both weight-equitable, Lemma 14 implies that $F(V, \mathcal{P})$ and $F(V, \mathcal{Q})$ are both \mathcal{B} -invariant. It then follows from [6, Lemma 2.3] that $F(V, \mathcal{P} \vee \mathcal{Q})$ is also \mathcal{B} -invariant, so $\mathcal{P} \vee \mathcal{Q}$ is weight-equitable. \square

As a result of Lemma 14 and Corollary 15, we obtain the equitable partition results from [6, Section 5]. Also, observe that as a consequence of Corollary 15, any partition \mathcal{P} has a unique maximal weight-equitable refinement, which is given by the join of all the weight-equitable partitions that refine \mathcal{P} . Contrary to the join, the meet of two weight-equitable partitions need not be weight-equitable, as illustrated in Example 16.

Example 16. Consider the graph G given in Fig. 2, which has Perron eigenvector $(1, \sqrt{2}, 1, 1, \sqrt{2}, 1)$. Partitions $\mathcal{P} = \{\{1, 3, 5\}, \{2, 4, 6\}\}$ and $\mathcal{Q} = \{\{1, 5, 6\}, \{2, 3, 4\}\}$ of G are weight-equitable, but not equitable. However, $\mathcal{P} \wedge \mathcal{Q} = \{\{1, 5\}, \{3\}, \{6\}, \{2, 4\}\}$ is not a weight-equitable partition.

In Section 5.3 we will show an application of the Corollary 15 for computing coarse weight-equitable partitions of cographs.

5. Computational aspects of weight-equitable partitions

As mentioned in the introduction, there are many cases in which one is interested in finding coarse weight-equitable partitions. While it is known that the coarsest equitable partition of a graph can be found in polynomial time, see for example Corneil et al. [7] and Bastert [4], nothing seems to be known about computing weight-equitable partitions. For the latter, Bastert's approach [4] becomes trivial, since the initialization step of his algorithm includes all vertices in the same cell, a case which is always weight-equitable (see Table 1). Thus, Bastert's algorithm cannot be generalized to weight-equitable partitions immediately.

The aim of this section is to investigate the potential of the join operator derived in the previous section in generating coarse partitions from finer ones. Moreover, a natural choice for a fine partition is to study partitions whose cells all have the same size c . We call such a partition c -homogeneous, where the parameter c controls the coarseness of the partition.

To study the potential of the join operator systematically, however, we need to be able to efficiently generate fine weight-equitable partitions. Since finding equitable 2-homogeneous partitions is NP-hard (Section 5.1), we focus in our investigation on the join operator on cographs. We will see that for such graphs the concept of equitability and weight-equitability coincides for 2-homogeneous partitions and that we can find (weight-) equitable 2-homogeneous partitions very efficiently (Section 5.2). Based on these results, Section 5.3 investigates the capability of the join operator to find coarse weight-equitable partitions.

5.1. Finding fine equitable partitions is hard

Let G be an undirected graph. An *automorphism* of G is a bijection $\gamma: V \rightarrow V$ that preserves adjacency, i.e., $\{\gamma(u), \gamma(v)\} \in E$ if and only if $\{u, v\} \in E$. An automorphism γ is *fixed-point-free* if there is no vertex v such that $\gamma(v) = v$. The *order* of γ is the smallest positive integer i such that γ^i is the identity. An automorphism of order two is called an *involution*.

Lubiw [24] studied the complexity of several algorithmic problems related to graph automorphism. In particular, she has shown that deciding whether a given graph G has a fixed-point-free automorphism of order two is NP-complete. With the following observation, we can link this to the complexity of finding certain equitable partitions.

Lemma 17. *Let G be an undirected graph. Then, G has an automorphism being an involution without fixed points if and only if G admits an equitable partition with $\frac{n}{2}$ cells each having size 2.*

Proof. Suppose G has an equitable partition $\{V_1, \dots, V_m\}$ with $m = \frac{n}{2}$ and $|V_i| = 2$ for all $i \in [m]$. For every $i \in [m]$, we assume that V_i is given by $\{i_1, i_2\}$. To prove the first part of the assertion, we show that $\gamma = \prod_{i=1}^m (i_1, i_2)$ is an automorphism of G that is an involution without fixed points. Since $\{V_1, \dots, V_m\}$ is a partition of V and all cells have cardinality 2, γ is an involution without fixed points. Hence, it remains to show that γ is an automorphism of G .

Let $i, j \in [m]$ and $r, s \in [2]$. Moreover, let r' and s' be the complementary indices of r and s in $[2]$, respectively. To show that γ is an automorphism, we need to show that $e = \{i_r, j_s\} \in E$ if and only if $\gamma(e) = \{i_{r'}, j_{s'}\} \in E$. Observe that both i_r and $i_{r'}$ have the same number of neighbors in U_j , since $\{V_1, \dots, V_m\}$ is equitable. Consequently, if i_r is not adjacent with U_j , also $i_{r'}$ is not, which implies $e, \gamma(e) \notin E$. Furthermore, if i_r is adjacent with j_s and $j_{s'}$, so is $i_{r'}$. Hence, $e, \gamma(e) \in E$. Finally, it remains to consider the case that i_r and $i_{r'}$ are adjacent with exactly one vertex in U_j . Then, both cannot be adjacent with the same vertex in U_j , because otherwise one vertex in U_j would have two neighbors in U_i while the other has no neighbor in U_i , contradicting equitability. This again implies $e \in E$ if and only if $\gamma(e) \in E$, concluding the first part of the proof.

For the reverse direction, let $\gamma = \prod_{i=1}^m (i_1, i_2)$ be an involutory automorphism of G without fixed points. Then $\{V_1, \dots, V_m\}$, where, for every $i \in [m]$, $V_i = \{i_1, i_2\}$, is a partition of V . As γ is an automorphism of G , each node in V_i has the same number of neighbors in V_j for all $i, j \in [m]$. Hence $\{V_1, \dots, V_m\}$ is an equitable partition. \square

Due to the aforementioned result by Lubiw [24], we thus conclude the following result.

Corollary 18. *Deciding whether a given graph G admits an equitable partition with $\frac{n}{2}$ cells is NP-complete.*

5.2. Finding weight-equitable partitions for cographs

Corollary 18 shows that, unless $P = NP$, we can generally not decide in polynomial time whether a graph admits an equitable partition with cells of size two. However, for certain graph families we can exploit the graph structure to obtain an efficient algorithm to compute such partitions. We will show that for the class of cographs the existence of these specific equitable partitions can be decided in polynomial time.

Definition 19 ([22]). A *cograph* is defined recursively using the following three rules:

- (i) a graph on a single vertex is a cograph;
- (ii) if G_1, \dots, G_k are cographs, then so is $G_1 \cup \dots \cup G_k$;
- (iii) if G is a cograph, then so is its complement \overline{G} .

Alternatively, an undirected simple graph is a cograph if and only if it does not contain an induced P_4 [23].

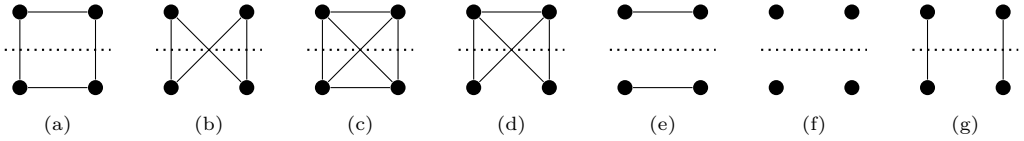


Fig. 3. All possible subgraphs induced by two cells of a weight-equitable partition of a cograph with cells of size two. The dotted line separates the two cells.

Cographs have been studied extensively in the literature. From the spectral point of view, Jung [21] introduced an algorithm for locating eigenvalues of cographs in a given interval. Ghorbani [12] provided a new characterization of cographs, and further properties of the eigenvalues (of the adjacency matrix) of a cograph were explored, e.g., by Ghorbani [11], Mohammadian and Trevisan [25], and Jacobs et al. [20].

Before we describe an algorithm to find 2-homogeneous (weight-) equitable partitions of a cograph G , we first show that there is indeed no difference between 2-homogeneous weight-equitable partitions and equitable partitions.

Lemma 20. *Let G be a connected cograph. Then, every weight-equitable partition with $\frac{n}{2}$ cells each having size 2 of G is an equitable partition of G .*

Proof. Suppose G has a weight-equitable partition $\{V_1, \dots, V_m\}$ with $m = \frac{n}{2}$ and $|V_i| = 2$ for every $i \in [m]$. For every $i \in [m]$, we assume that V_i is given by $\{i_1, i_2\}$. Consider two arbitrary cells V_i and V_j and their induced subgraph. Note that if i_1 has no neighbors in V_j , then neither does i_2 and vice versa. As we show next, i_1 and i_2 must have the same number of neighbors in V_j .

Without loss of generality, assume that i_1 is adjacent to both j_1 and j_2 and i_2 only to j_1 . If $i_1 \sim i_2$, then weight-equitability ensures that $\nu_{i_1} = \nu_{i_2}$, as $b_{ii}^*(i_1) = b_{ii}^*(i_2)$. At the same time, it must hold that $b_{ij}^*(i_1) = b_{ij}^*(i_2)$. This implies that $\nu_{j_1} = \nu_{j_1} + \nu_{j_2}$, contradicting the fact that $\nu > 0$. If $i_1 \not\sim i_2$ and $j_1 \not\sim j_2$, then V_i and V_j induce a path of length three, which contradicts the fact that G is cograph. The case $j_1 \sim j_2$ is symmetric to $i_1 \sim i_2, j_1 \not\sim j_2$. This means that vertices that share a cell have an equal number of neighbors in every other cell.

Fig. 3 shows (up to symmetry) all possible induced subgraphs of two cells that satisfy the above requirement. In graph (a)-(e), weight-equitability directly implies that the Perron eigenvector is constant over both cells. We will show that this is also the case for (f) and (g). By Proposition 6, a weight-equitable partition with constant Perron entries over each part is equitable, so the result follows.

Assume that V_i and V_j induce the empty subgraph. Since G is connected, there must be some part $V_k \neq V_j$ such that $G[V_i \cup V_k]$ is not empty, so we may instead consider one of the other subgraphs to determine ν on V_i . If V_i and V_j induce subgraph (g), consider a shortest path between either endpoint of edge $\{i_1, j_1\}$ and $\{i_2, j_2\}$. Fig. 4 shows the general shape of the graph induced by this path and V_i, V_j , assuming without loss of generality that i_1 and i_2 are the endpoints of the path. Note that there always exists an

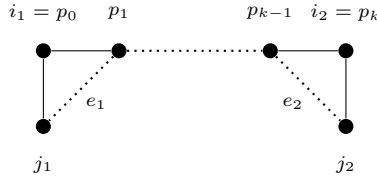


Fig. 4. The subgraph induced by (g) and a shortest path between its two edges.



(a) A cograph with an equitable partition of 2-cells. Any refinement of its partition is again equitable.

(b) A cograph with a 3-balanced weight-equitable partition, which is not equitable.

Fig. 5. Examples of cographs.

induced path of length three, unless both edges e_1 and e_2 exist and $p_1 = p_{k-1}$. Since G is a cograph, we only need to consider the latter case. Let v be the vertex which forms a cell with p_1 . This vertex must have the same number of neighbors in V_i , so it is adjacent to i_1 and i_2 . Then V_i and $\{p_1, v\}$ induce either case (b) or (d), hence v is constant over V_i . \square

As equitable partitions are always weight-equitable, we obtain the following corollary.

Corollary 21. *Let G be a connected cograph. Then, G has an automorphism being an involution without fixed points if and only if G admits a (weight-) equitable partition with $\frac{n}{2}$ cells each having size 2.*

Proof. The result follows directly from Lemma 17 and Lemma 20. \square

Note that partitions with cells of size two are not necessarily the finest equitable partitions of cographs. Fig. 5a shows an example of a cograph and an equitable 2-homogeneous partition which is not the finest. In fact, one can verify that any refinement of this partition is also equitable. Moreover, Lemma 20 cannot be generalized to c -homogeneous partitions with $c > 2$ as the cograph in Fig. 5b admits a weight-equitable partition which is not equitable.

We now turn the focus back to finding 2-homogeneous weight-equitable partitions of cographs. To devise an algorithm finding such partitions, we consider the twin classes of these graphs.

In a graph G , vertices u and v are called *twins* if $G(u) \setminus \{u, v\} = G(v) \setminus \{u, v\}$. This defines an equivalence relation on $V(G)$, each class of which is called a *twin class*. Cographs can be characterized by their twin pairs as follows.

Lemma 22 ([5], Theorem 11.3.3). *A graph G is a cograph if and only if every induced subgraph of G with more than one vertex has a pair of twins.*

Note that twin classes, like equitable partitions, are based on similarities in the neighborhoods of vertices. However, twin vertices have identical neighborhoods, whereas vertices sharing a cell in an equitable partition are only required to have the same number of neighbors in every other cell. In the following theorem, we show that, for 2-homogeneous partitions, these concepts are related nevertheless.

Lemma 23. *Let G be a graph with a 2-homogeneous equitable partition \mathcal{P} . If $\{u, v\}$ and $\{u', w\}$ are two cells of \mathcal{P} such that u, u' are twins, then v, w are also twins.*

Proof. We show that for every cell $\{x, y\}$ of \mathcal{P} , $G(v) \setminus \{x, y\} = G(w) \setminus \{x, y\}$. From this, follows that v, w are indeed twins.

Let $|G(v) \cap \{x, y\}| = \ell$. Since u, v belong to the same cell and u, u' are twins, we know that

$$\ell = |G(v) \cap \{x, y\}| = |G(u) \cap \{x, y\}| = |G(u') \cap \{x, y\}|.$$

Then $|G(w) \cap \{x, y\}| = \ell$, as u', w belong to the same cell. This means that we are done if $\ell = 0, 2$.

Assume that $\ell = 1$ and, without loss of generality, $v \sim y$. In that case, we must have $|G(x) \cap \{u, v\}| = |G(y) \cap \{u, v\}| = 1$, and hence $u \sim x$. As u, u' are twins, we have $G(u') \cap \{x, y\} = \{x\}$. This in turn implies that $G(w) \cap \{x, y\} = \{y\}$, that is $w \sim y$, and so we are done. A similar argument holds when $\{x, y\} = \{u, v\}$ or $\{u', w\}$. \square

Theorem 24. *Let G be a cograph, then the following are equivalent:*

- (i) *G has an equitable partition in which every cell has size two;*
- (ii) *G has an equitable partition in which every cell consists of a pair of twins;*
- (iii) *the size of every twin class of G is even.*

Proof. The implications (ii) \Rightarrow (i) and (ii) \Leftrightarrow (iii) are clear. We prove (i) \Rightarrow (ii) by induction on n , the order of G . The statement is trivially true for $n = 2$. Let $n \geq 4$ and let \mathcal{P} be an equitable partition of G given by (i). By Lemma 22, G has a pair of twins, say $\{u, u'\}$.

First assume that $\{u, u'\} \in \mathcal{P}$. Then $\mathcal{P} \setminus \{\{u, u'\}\}$ is an equitable partition for the subgraph $G' := G \setminus \{u, u'\}$. This means that G' satisfies (i) and so by the induction hypothesis, it satisfies (ii). Let \mathcal{P}' be an equitable partition of G' in which every cell is a pair of twins. Twins in G' remain twins in G , as G' is obtained by removing a pair of twins from G . It follows that $\mathcal{P}' \cup \{\{u, u'\}\}$ is a desired partition for G .

Now assume that $\{u, u'\} \notin \mathcal{P}$. Then $\{u, v\}, \{u', w\} \in \mathcal{P}$ for some vertices v, w . By Lemma 23, the vertices v, w are twins. Thus, $\mathcal{P} \setminus \{\{u, v\}, \{u', w\}\}$ is an equitable partition

for $G' := G \setminus \{u, u', v, w\}$. Hence G' satisfies (i) and so by the induction hypothesis, it satisfies (ii). Let \mathcal{P}' be an equitable partition of G' in which every cell is a pair of twins. Twins in G' remain twins in G since G' is obtained by removing two pairs of twins from G . Therefore, $\mathcal{P}' \cup \{\{u, u'\}, \{v, w\}\}$ is a desired partition for G . \square

Note that not every 2-homogeneous equitable partition arises from twin pairs. Consider for example the cycle C_4 and a partition induced by a perfect matching. However, if a graph admits a 2-homogeneous equitable partition, Theorem 24 implies that it has such a partition where each cell forms a twin pair. In C_4 , pairing the antipodal vertices gives the required partition.

Using Theorem 24 we can explicitly calculate 2-homogeneous partitions of a cograph G in polynomial time. The twin classes can be found in $O(|V(G)|^2)$ time by hashing the neighborhood of every vertex and comparing them pairwise. We know that these classes all must have even size. Splitting them (arbitrarily) into pairs then gives a 2-homogeneous equitable partition. Combining the previous results from this section, we obtain our last main result.

Theorem 25. *Let G be a cograph. The problem of deciding whether G admits a (weight-) equitable partition with $\frac{n}{2}$ cells of size 2 can be solved in $O(n^2)$ time.*

The above computation method can be adapted to a more general setting. To find c -homogeneous partitions for any c , we can look for twin classes whose size is a multiple of c . It is easy to see that the implications (ii) \Leftrightarrow (iii) in Theorem 24 still hold if we replace ‘pair’ and ‘even’ by ‘ c -set’ and ‘a multiple of c ’. To complete the more general algorithm, we also need to show the c -homogeneous equivalent of (i) \Rightarrow (ii), i.e., if G admits an equitable c -homogeneous partition, it admits one such that every cell consists of twins. However, this need not be the case, as illustrated by Example 26. This means that when our method is successful, a c -homogeneous equitable partition is guaranteed to exist, but it may produce false negatives.

Example 26. Consider the three-dimensional cube. Each of its 4-homogeneous equitable partitions can be formed by considering a perfect matching and grouping the matching edges into two pairs. However, in such partitions, vertices which share a cell are never twins.

5.3. Joins of weight-equitable partitions

In Section 4, it was shown that the join of two weight-equitable partitions is again weight-equitable. If we have a number of weight-equitable partitions at hand, this gives us a method to construct coarser ones. An interesting question is, on the one hand, how close one can get to the coarsest nontrivial partition with a small number of join operations. On the other hand, one may wonder how many joins can be done before

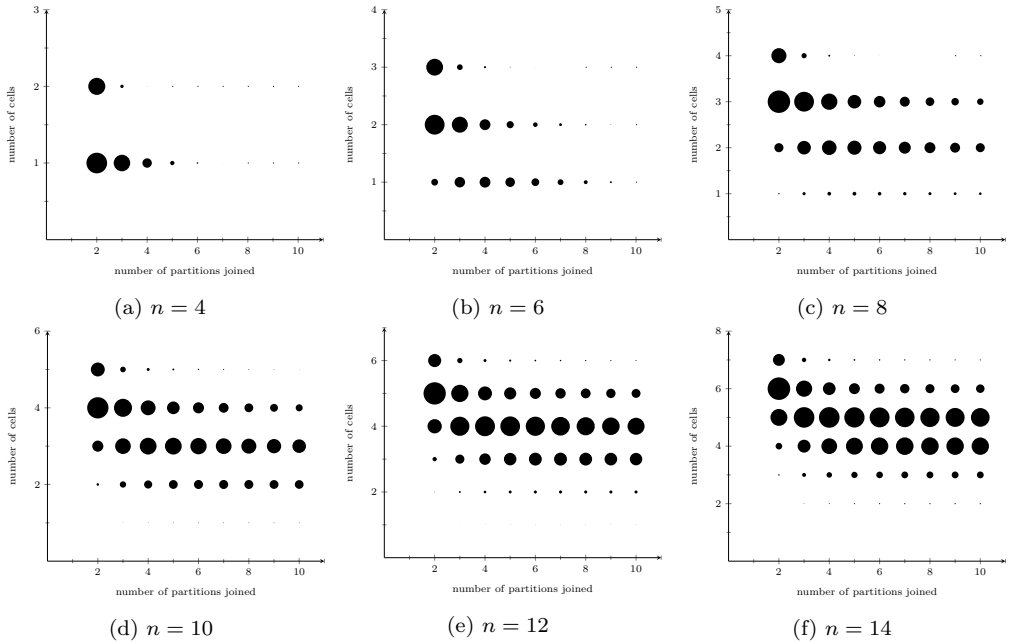


Fig. 6. Distribution of coarseness with respect to the number of join operations for small graphs.

obtaining the trivial partition. In this section, we study these questions empirically for cographs, using Theorem 24 to construct the initial weight-equitable partitions. Recall that for cographs, weight-equitable partitions and equitable partitions coincide, hence Bastert’s algorithm [4] finds the coarsest (weight-) equitable partition in polynomial time. However, for general graphs no algorithm is known and the join operation may provide a useful approximation method.

The setup is the following. For a small even integer n , generate all connected cographs on n vertices. For each cograph G , compute its twin classes to determine whether it admits a 2-homogeneous partition. If not, it is discarded, and if it does, a 2-homogeneous partition is found. As n is small, the generators $g_1 \dots, g_k$ of $\text{Aut}(G)$ can be computed quickly. Nine additional 2-homogeneous partitions are sampled by generating $m_i \in [\text{ord}(g_i)]$ and applying the automorphism $\prod_{i=1}^k g_i^{m_i}$ to the given partition. We take the join of every possible subset of the ten 2-homogeneous partitions and count the number of cells of the resulting weight-equitable partitions.

Fig. 6 summarizes the results for $n = 4, 6, \dots, 14$. The size of the circle at position (x, y) represents how often joining x partitions results in a partition with y cells. Every column is normalized by the number of ways to choose x partitions. Note that for $n = 4$ and $n = 6$ the last columns are relatively sparse, because many cographs of this size do not admit ten distinct 2-homogeneous partitions. For the larger graphs, it is most likely to obtain three to five cells by merging ten partitions. In very few cases it results in the trivial partition.

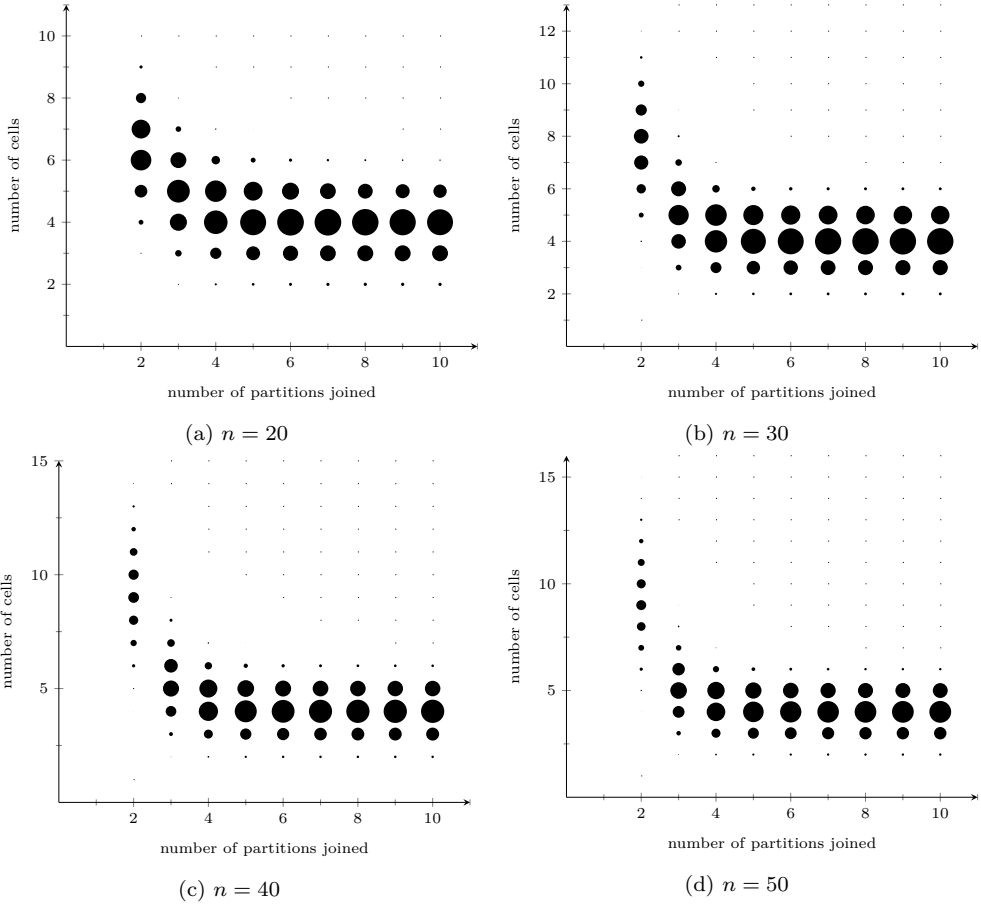


Fig. 7. Distribution of coarseness with respect to the number of join operations.

To see whether these patterns continue for larger graphs, we repeat the procedure for 200 cographs on 20, 30, 40 and 50 vertices which admit a 2-homogeneous weight-equitable partition. In this case, merging ten partitions most likely gives a partition with four cells. However, four partitions are already enough to obtain a similar distribution (Fig. 7).

We conclude that using the join operation discussed in Corollary 15 for just a few times is able to generate very coarse partitions even if we start with (weight-) equitable partitions that are very fine. Thus, although we do not know how to compute a coarsest non-trivial weight-equitable partition, the join operator allows us to get a reasonably good approximation.

Declaration of competing interest

None declared.

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