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DECOMPOSITION THEOREMS FOR INVOLUTIVE SOLUTIONS TO THE YANG–BAXTER EQUATION

S. RAMÍREZ, L. VENDRAMIN

ABSTRACT. Motivated by the proof of Rump of a conjecture of Gateva–Ivanova on the decomposability of square-free solutions to the Yang–Baxter equation, we present several other decomposability theorems based on the cycle structure of a certain permutation associated with the solution.

1. INTRODUCTION

In [11] Drinfeld considered set-theoretic solutions, i.e. pairs (X, r) , where X is a non-empty set and $r: X \times X \rightarrow X \times X$ is a bijective map such that

$$(r \times \text{id})(\text{id} \times r)(r \times \text{id}) = (\text{id} \times r)(r \times \text{id})(\text{id} \times r).$$

The first systematic account about these solutions was written by Etingof, Schedler and Soloviev [12] and Gateva-Ivanova and Van den Bergh [16]. Both papers studied involutive solutions. The theory of such solutions was further developed in several other papers, see for example [4, 6, 7, 13–15, 20, 21].

Originally Gateva–Ivanova mainly studied involutive square-free solutions, i.e. solutions (X, r) such that $r(x, x) = (x, x)$ holds for all $x \in X$. Square-free solutions produce algebraic structures with several interesting properties, see for example [13]. Nowadays it is clear that not only square-free solutions are of interest. In fact, several intriguing connections involving arbitrary involutive solutions were found recently [22]. These connections include radical rings [21], homology [18], ordered and diffuse groups [3, 9, 19] and Garside theory [8, 10].

Recent research on the combinatorics of the Yang–Baxter equation is mainly based on the use of powerful algebraic methods. This leads to several breakthroughs in the area and new families of solutions were found. However, the problem of constructing new classes of solutions is still challenging and a program to classify all solutions is somewhat still out of reach. There are two promising approaches in this context. One is based on *indecomposable* solutions and the other one is based on *retractable* solutions. In both cases, the idea is to understand small solutions suitable to play the role of building blocks for the theory.

The starting point of this work is an old conjecture of Gateva–Ivanova about the decomposability of certain solutions. In a 1996 talk at the *International Algebra Conference* in Hungary, she conjectured that finite *square-free* solutions are always decomposable, that is solutions where the underlying set X that can be decomposed into a disjoint union $X = Y \cup Z$ of subsets Y and Z of X such that $r(Y \times Y) \subseteq Y \times Y$ and $r(Z \times Z) \subseteq Z \times Z$.

The conjecture was proved by Rump in [20]. In the same paper he also showed that the conjecture cannot be extended to infinite solutions.

Based on the work of Etingof, Schedler and Soloviev [12], we consider the diagonal of a solution (X, r) , a certain permutation T of the set X . The cycle structure of this permutation turns out to be an invariant of the solution. Rump's theorem can then be restated as follows: If the diagonal map of a solution (X, r) fixes all points of X , then (X, r) is decomposable.

In this work we deal with decomposition theorems similar to that of Rump. The basic tools are the diagonal map of a solution and a recent result of Cedó, Jespers and Okniński about solutions with a primitive permutation group [5]. It seems that Rump's result is just the tip of the iceberg, as—at least for finite solutions—the bijective map T turns out to encode deep combinatorial information about the structure of the solution. In particular, the map T could be used to detect (in)decomposability or retractability of solutions.

The paper is organized as follows. We recall some preliminaries in Section 2. We recall the classification of involutive solutions with a prime number of elements found by Etingof, Schedler and Soloviev in [12], Rump's theorem on decomposability of square-free solutions [20] and the theorem of Cedó, Jespers and Okniński on involutive solutions with primitive permutation groups [5]. Our results appear in Section 3. In Theorem 3.2 we prove that indecomposable solutions are always defined by permutations different from the identity. We remark that this result might be useful for the problem of constructing explicitly indecomposable solutions of small size. In Theorem 3.5 we prove that involutive solutions where the diagonal is a cycle of maximal length are indecomposable. Theorem 3.10 deals with the decomposability of those solutions where the diagonal fixes only one point. Theorem 3.13 is devoted to the decomposability of those solutions where the diagonal fixes two or three points. We conclude the paper with some questions and conjectures.

preliminaries

2. PRELIMINARIES

A *set-theoretic solution to the Yang–Baxter equation* is a pair (X, r) , where X is a non-empty set and $r: X \times X \rightarrow X \times X$ is a bijective map such that

$$(r \times \text{id})(\text{id} \times r)(r \times \text{id}) = (\text{id} \times r)(r \times \text{id})(\text{id} \times r).$$

By convention, we write

$$r(x, y) = (\sigma_x(y), \tau_y(x)).$$

A solution (X, r) is said to be *non-degenerate* if all the maps $\sigma_x: X \rightarrow X$ and $\tau_x: X \rightarrow X$ are bijective for all $x \in X$, and it is said to be *involutive* if $r^2 = \text{id}_{X \times X}$. Note that for non-degenerate involutive solutions,

$$\tau_y(x) = \sigma_{\sigma_x(y)}^{-1}(x) \quad \text{and} \quad \sigma_x(y) = \tau_{\tau_y(x)}^{-1}(y).$$

By convention, a *solution* will be a non-degenerate involutive solution to the Yang–Baxter equation.

A *homomorphism* between the solutions (X, r_X) and (Y, r_Y) is a map $f: X \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{r_X} & X \times X \\ f \times f \downarrow & & \downarrow f \times f \\ Y \times Y & \xrightarrow{r_Y} & Y \times Y \end{array}$$

is commutative, i.e. $(f \times f) \circ r_X = r_Y \circ (f \times f)$. An isomorphism of solutions is a bijective homomorphism of solutions. Solutions form a category.

A solution (X, r) is said to be *square-free* if $r(x, x) = (x, x)$ for all $x \in X$.

In [12, Proposition 2.2] it is proved that if (X, r) is a solution, then the map $T: X \rightarrow X$, $T(x) = \tau_x^{-1}(x)$, is invertible with inverse $x \mapsto \sigma_x^{-1}(x)$ and, moreover,

$$\tau_x^{-1} \circ T = T \circ \sigma_x$$

for all $x \in X$.

Definition 2.1. The *diagonal* of a solution (X, r) is defined as the permutation $T: X \rightarrow X$, $x \mapsto \tau_x^{-1}(x)$.

The *permutation group* of a solution (X, r) is defined as the subgroup $\mathcal{G}(X, r)$ of Sym_X generated by the set $\{\sigma_x : x \in X\}$. Note that if X is finite, then $\mathcal{G}(X, r)$ is finite. The group $\mathcal{G}(X, r)$ naturally acts on X . A solution (X, r) is said to be *indecomposable* if the group $\mathcal{G}(X, r)$ acts transitively on X and *decomposable* otherwise. This definition of decomposability turns out to be equivalent to the definition mentioned in the introduction, see for example the proof of [12, Proposition 2.12]. Using the invertible map T one proves that $\mathcal{G}(X, r)$ is isomorphic (as a permutation group on the set X) to the group generated by the set $\{\tau_x : x \in X\}$.

thm:ESS

Theorem 2.2 (Etingof–Schedler–Soloviev). *Let p be a prime number. An indecomposable solution with p elements is a cyclic solution, i.e. a solution isomorphic to $(\mathbb{Z}/p, r)$, where $r(x, y) = (y - 1, x + 1)$.*

Proof. See [12, Theorem 2.13] □

We recall Rump’s theorem.

thm:Rump

Theorem 2.3 (Rump). *Let (X, r) be a finite solution. If the diagonal of the solution is the identity, then (X, r) is decomposable.*

Proof. See [20, Theorem 1]. □

A solution (X, r) is said to be *primitive* if the group $\mathcal{G}(X, r)$ acts primitively on X . In the Oberwolfach mini-workshop *Algebraic Tools for Solving the Yang–Baxter Equation*, Ballester–Bolinches posed the following conjecture [17]: Finite primitive solutions are of size p for some prime number p . The conjecture was proved by Cedó, Jespers and Okniński. The proof uses the theory of braces and Theorem 2.2.

thm:CJO

Theorem 2.4 (Cedó–Jespers–Okniński). *If (X, r) is a finite primitive solution, then $|X|$ is a prime number and (X, r) is a cyclic solution.*

Proof. See [5, Theorem 3.1]. □

In order to prove the conjecture mentioned in the introduction, Rump introduced a better way to deal with solutions. A *cycle set* is a non-empty set X provided with a binary operation $X \times X \rightarrow X$, $(x, y) \mapsto x \cdot y$, such that the maps $\varphi_x: X \rightarrow X$, $y \mapsto x \cdot y$, are bijective for all $x \in X$ and

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

holds for all $x, y, z \in X$.

A *homomorphism* between the cycle sets X and Z is a map $f: X \rightarrow Z$ such that $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in X$. An *isomorphism* of cycle sets is a bijective homomorphism of cycle sets. Cycle sets form a category.

A cycle set X is said to be *non-degenerate* if the map $X \rightarrow X$, $x \mapsto x \cdot x$, is bijective. Rump proved that solutions are in bijective correspondence with non-degenerate cycle sets. The correspondence is given by

$$r(x, y) = ((y * x) \cdot y, y * x),$$

where $y * x = z$ if and only if $x = y \cdot z$, see [20, Proposition 1]. Moreover, the category of solutions and the category of non-degenerate cycle sets are equivalent.

The permutation group $\mathcal{G}(X)$ of a cycle set X is defined as the permutation group of its associated solution. Indecomposable and decomposable cycle sets are then defined in the usual way.

The diagonal of a cycle set X is then defined as the diagonal of its associated solution. Clearly, the cycle structure of the diagonal of a finite cycle set is invariant under isomorphisms. Note that a solution is square-free if and only if the diagonal of its associated cycle set is the identity.

We write Sym_X to denote the set of bijective maps $X \rightarrow X$.

lem:diagonal

Lemma 2.5. *Let X be a cycle set with diagonal T . If $U: X \rightarrow X$ is a bijective map and U and T are conjugate in Sym_X , then there exists an isomorphic cycle set structure on the set X with diagonal U .*

Proof. Let $(x, y) \mapsto x \cdot y$ be the cycle set operation on X and let $\gamma \in \text{Sym}_X$ be such that $U = \gamma^{-1} \circ T \circ \gamma$. A direct calculation shows that the operation $y \cdot_U z = \gamma^{-1}(\gamma(y) \cdot \gamma(z))$ turns X into a cycle set isomorphic to the original X . Moreover,

$$y \cdot_U y = \gamma^{-1}(\gamma(y) \cdot \gamma(y)) = \gamma^{-1}(T(\gamma(y))) = (\gamma^{-1} \circ T \circ \gamma)(y) = U(y)$$

holds for all $y \in X$. □

theorems

3. DECOMPOSITION THEOREMS

For the first results in this section we use the language of cycle sets. We start with some easy consequences of Theorem 2.4 and the results of [2].

thm:decomposable

Theorem 3.1. *Let X be a finite cycle set with more than one element. If some φ_x is a cycle of length at least $|X|/2$ and is coprime with $|X|$ then X is decomposable.*

Proof. We first note that Theorem 2.4 implies that in none of these cases $\mathcal{G}(X)$ can act primitively on X . Let y be a fixed point of φ_x and B a non-trivial block containing y . Since φ_x fixes y we must have $\varphi_x(B) = B$. If this block were to contain an element that is not fixed by φ_x , and since this

permutation is a cycle, it would have to contain all of them, which cannot happen since the block would then have more than $|X|/2$ elements. Since y was arbitrary this means any block containing a point fixed by φ_x must consist entirely of fixed points. In particular the size of the block must divide the number of fixed points. Since it also has to divide $|X|$ and these numbers are coprime, there cannot be non-trivial blocks, contradicting the imprimitivity of the solution. \square

thm:no_id

Theorem 3.2. *If X is a finite cycle set with more than one element and there exists some $x \in X$ such that $\varphi_x = \text{id}$, then X is decomposable.*

Proof. Let $x \in X$ be such that $\varphi_x = \text{id}$. Then

$$y \cdot z = (x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

holds for all $y, z \in X$. This implies that $\varphi_{y \cdot x} = \text{id}$ for all $y \in X$ and hence $\varphi_{g \cdot x} = \text{id}$ for all $g \in \mathcal{G}(X, r)$. Let $Y = \{x \in X : \varphi_x = \text{id}\}$. If $Y = X$, then the cycle set is trivial and hence it is decomposable. If not, then the group $\mathcal{G}(X)$ acts on Y . Thus Y is a non-trivial $\mathcal{G}(X)$ -orbit and hence X is decomposable. \square

Let us show some direct consequences of the theorem. A cycle set X is said to be *retractable* if the equivalence relation on X given by

$$x \sim y \iff \varphi_x = \varphi_y$$

is non-trivial. This notion corresponds to that of *retractable solutions*.

cor:regular

Corollary 3.3. *Let X be a finite cycle set such that $|X| > 1$. If $\mathcal{G}(X)$ acts regularly on X , then X is retractable.*

Proof. If $\mathcal{G}(X)$ acts regularly on X , then $|\mathcal{G}(X)| = |X|$. Since X is indecomposable, $\varphi_x \neq \text{id}$ for all $x \in X$ by Theorem 3.2. Then some φ_x must appear at least twice as a defining permutation of X . \square

The following consequence goes back to Rump, see [20, Proposition 1].

Corollary 3.4. *If X is an indecomposable finite cycle set with more than one element such that $\mathcal{G}(X)$ is abelian, then X is retractable.*

Proof. The assumptions imply that $\mathcal{G}(X)$ acts regularly on X . Thus the claim follows from Corollary 3.3. \square

Now we present several results similar to that of Rump about decomposability of square-free solutions. First we consider the case where the diagonal of the solution is a cycle of maximal length.

thm:n

Theorem 3.5. *Let X be a cycle set of size n . If T is a cycle of length n , then X is indecomposable.*

Proof. Let $x, y \in X$. It is enough to prove that if $m \in \mathbb{N}$ and $T^m(x) = y$, then x and y belong to the same $\mathcal{G}(X)$ -orbit. This is clearly true if $m = 1$. Now assume that the result holds for some $m \geq 1$, then

$$y = T^{m+1}(x) = T(T^m(x))$$

implies that $T^m(x)$ and y belong to the same $\mathcal{G}(X)$ -orbit. Then the inductive hypothesis implies the claim. \square

In Theorem 3.5, it is important that the diagonal is a cycle of maximal length. A diagonal that moves all points does not necessarily imply indecomposability.

exa:581

Example 3.6. Let $X = \{1, \dots, 6\}$ and $r(x, y) = (\sigma_x(y), \tau_y(x))$ be given by

$$\begin{aligned} \sigma_1 &= (123), & \sigma_2 &= (123)(456), & \sigma_3 &= (123)(465), & \sigma_4 &= \sigma_5 = \sigma_6 = (456), \\ \tau_1 &= (132), & \tau_2 &= (132)(465), & \tau_3 &= (132)(456), & \tau_4 &= \tau_5 = \tau_6 = (465). \end{aligned}$$

Then (X, r) is decomposable and $T = (123)(456)$ moves all points of X .

We shall need the following lemmas.

lem:T

Lemma 3.7. *Let (X, r) be a solution and $x, y \in X$. Then $T(x) = y$ if and only if $r(y, x) = (y, x)$.*

Proof. We first compute

$$r(T(x), x) = (\sigma_{T(x)}(x), \tau_x T(x)) = (\sigma_{T(x)}(x), x).$$

Since (X, r) is involutive, it follows that $y = T(x) = \sigma_{T(x)}(x)$. Conversely, if $r(y, x) = (y, x)$, then $\tau_x(y) = x$ and hence $y = T(x)$. \square

The following result is well-known. It follows from a well-known general formula.

lem:Tx=x

Lemma 3.8. *Let (X, r) be a solution and $x \in X$ be such that $T(x) = x$. Then $T(\tau_y(x)) = \tau_{\sigma_x(y)}(x)$ for all $y \in X$.*

Proof. The Yang–Baxter equation applied to the tuple (x, x, y) yields

$$r_2(\sigma_x \sigma_x(y), \tau_{\sigma_x(y)}(x), \tau_y(x)) = (\sigma_x \sigma_x(y), \tau_{\sigma_x(y)}(x), \tau_y(x)).$$

Thus $r(\tau_{\sigma_x(y)}(x), \tau_y(x)) = (\tau_{\sigma_x(y)}(x), \tau_y(x))$. Now Lemma 3.7 implies that $T(\tau_y(x)) = \tau_{\sigma_x(y)}(x)$. \square

Applying the lemma to square-free solutions we get that square-free cycle sets are balanced. This result was noted by Rump, see Definition 2 and the following paragraph of [20]. We provide a proof for completeness.

Proposition 3.9. *If (X, r) is a square-free solution, then $\tau_x = \sigma_x^{-1}$ for all $x \in X$.*

Proof. Let $x, y \in X$. We know that all solutions satisfy $\tau_x(y) = \sigma_{\sigma_y(x)}^{-1}(y)$. Since (X, r) is square-free, $T(y) = y$ and $T(\tau_x(y)) = \tau_x(y)$. Lemma 3.8 implies that $\tau_{\sigma_y(x)}(y) = \tau_x(y) = \sigma_{\sigma_y(x)}^{-1}(y)$, from where the claim follows. \square

We consider the case where the diagonal fixes one point.

thm:fix1

Theorem 3.10. *Let (X, r) be a finite solution of size $n > 1$. If the diagonal of (X, r) is a cycle of length $n - 1$, then X is decomposable.*

Proof. Let $x_0 \in X$ be such that $T(x_0) = x_0$ and $Y = X \setminus \{x_0\}$. Assume that (X, r) is indecomposable. Let $f: Y \rightarrow X$ be given by $y \mapsto \tau_y(x_0)$. Since (X, r) is indecomposable, there exists $y \in Y$ such that τ_y moves x_0 , as $x_0 = T(x_0) = \tau_{x_0}(x_0)$ and $\mathcal{G}(X, r) = \langle \tau_y : y \in Y \rangle$. So let $y \in Y$ be such that $f(y) \in Y$. Since T is a cycle of length $n - 1$, the permutation T moves each element of Y . This implies that

$$Y = \{f(y), T(f(y)), \dots, T^{n-2}(f(y))\}.$$

Since

$$T(f(y)) = T(\tau_y(x_0)) = \tau_{\sigma_{x_0}(y)}(x_0) = f(\sigma_{x_0}(y))$$

by Lemma 3.7, it follows that

$$Y = \{f(y), f(\sigma_{x_0}(y)), \dots, f(\sigma_{x_0}^{n-2}(y))\}.$$

Then σ_{x_0} permutes every element of Y cyclically.

We claim that $\mathcal{G}(X, r)$ acts primitively on X . If not, there exists a non-trivial block B . Without loss of generality we may assume that $x_0 \in B$. Then there exists $y \in Y$ such that $y \in B$. Since $\sigma_{x_0}(x_0) = x_0$, it follows that $\sigma_{x_0}(B) = B$. In particular, $\sigma_{x_0}(y) \in B$ and hence $Y \subseteq B$ because σ_{x_0} is a cyclic permutation on Y , a contradiction.

Since $\mathcal{G}(X, r)$ acts primitively on X , Theorem 2.4 implies that (X, r) is a cyclic solution. This is a contradiction, as the diagonal of a cyclic solution has no fixed points. \square

By inspection, solutions of size ≤ 10 such that the diagonal has only one fixed point are decomposable. This is not necessarily true if the diagonal fixes more than one point.

Example 3.11. Let $X = \{1, 2, \dots, 8\}$ and $r(x, y) = (\sigma_x(y), \tau_y(x))$, where

$$\begin{aligned} \sigma_1 &= (45), & \sigma_2 &= (36), & \sigma_3 &= (27), & \sigma_4 &= (18), \\ \sigma_5 &= (13428657), & \sigma_6 &= (17568243), & \sigma_7 &= (12468753), & \sigma_8 &= (13578642), \\ \tau_1 &= (46), & \tau_2 &= (35), & \tau_3 &= (28), & \tau_4 &= (17), \\ \tau_5 &= (18657243), & \tau_6 &= (13427568), & \tau_7 &= (13687542), & \tau_8 &= (12457863). \end{aligned}$$

Then $T = (57)(68)$ has four fixed points and (X, r) is indecomposable.

Example 3.12. Let $X = \{1, 2, \dots, 9\}$ and $r(x, y) = (\sigma_x(y), \tau_y(x))$, where

$$\begin{aligned} \sigma_1 &= (167925483), & \sigma_2 &= (125983467), & \sigma_3 &= (165923487), \\ \sigma_4 &= (158936472), & \sigma_5 &= (136972458), & \sigma_6 &= (156932478), \\ \sigma_7 &= (149)(268), & \sigma_8 &= (149)(357), & \sigma_9 &= (268)(357), \\ \tau_1 &= (185639742), & \tau_2 &= (189632745), & \tau_3 &= (139642785), \\ \tau_4 &= (198623754), & \tau_5 &= (128653794), & \tau_6 &= (194628753), \\ \tau_7 &= (295)(384), & \tau_8 &= (176)(295), & \tau_9 &= (176)(384). \end{aligned}$$

Then $T = (123)(456)$ has three fixed points and (X, r) is indecomposable.

Now we consider the case where the diagonal fixes two or three points.

thm:n-2_n-3

Theorem 3.13. Let (X, r) be a finite solution of size n .

- (1) If T is an $(n-2)$ -cycle and n is odd, then X is decomposable.
- (2) If T is an $(n-3)$ -cycle and $\gcd(3, n) = 1$, then X is decomposable.

Proof. Let m be the length of T ,

$$Y = \{y \in X : T(y) \neq y\} \quad \text{and} \quad Z = \{z \in X : T(z) = z\}.$$

If $X = Y \cup Z$ is a decomposition of X , there is nothing to prove. If not, there exists $z \in Z$ and $x \in X$ such that $\tau_x(z) \in Y$. Since T permutes the

elements of Y cyclically, the elements $\tau_x(z), T(\tau_x(z)), \dots, T^{m-1}(\tau_x(z))$ are all distinct. Moreover, since z is a fixed point,

$$\{\tau_x(z), T(\tau_x(z)), \dots, T^{m-1}(\tau_x(z))\} = \{\tau_x(z), \tau_{\sigma_z(x)}(z), \dots, \tau_{\sigma_z^{m-1}(x)}(z)\}.$$

In particular, $x, \sigma_z(x), \dots, \sigma_z^{m-1}(x)$ are all distinct elements, so the decomposition of σ_z into disjoint cycles contains a cycle of length at least m . Since T is a cycle of length m , $T^{k+m}(\tau_x(z)) = T^k(\tau_x(z))$, i.e.

$$\tau_{\sigma_z^{k+m}(x)}(z) = \tau_{\sigma_z^k(x)}(z).$$

Moreover, m is the minimal period which implies that the length of this cycle in σ_z has to be divisible by m . By the hypothesis on the m we are taking this can only happen if the length of the cycle is exactly m .

In the case where $m = n - 2$, since σ_z has at least one fixed point z , σ_z has to be a cycle of length m . By Theorem 3.1 X must be decomposable. In the case that $m = n - 3$ there are two possibilities, either σ_z is a cycle of length $n - 3$, and we can conclude the same way as before, or it is a product of cycle of length $n - 3$ and a cycle of length 2. If it is a product of two cycles we take a non-trivial block containing z . Like before this block cannot contain an element of the cycle of length $n - 3$. If it does not but it contains an element that belongs to the cycle of length 2 then it also has to contain the other element of the cycle. This means the block has size 3 and this is not possible because $\gcd(3, n) = 1$. \square

The assumption on n cannot be removed from the first claim of Theorem 3.13.

exa:T=12

Example 3.14. Let $X = \{1, 2, 3, 4\}$ and $r(x, y) = (\sigma_x(y), \tau_y(x))$, where

$$\begin{aligned} \sigma_1 &= (34), & \sigma_2 &= (1324), & \sigma_3 &= (1423), & \sigma_4 &= (12), \\ \tau_1 &= (24), & \tau_2 &= (1432), & \tau_3 &= (1234), & \tau_4 &= (13). \end{aligned}$$

Then $T = (12)$ and (X, r) is indecomposable.

Indecomposable solutions of small size are known up to size 11 and there are only 172 such solutions, see Table 3.1 [1].

TABLE 3.1. Indecomposable solutions of size $n \leq 11$.

n	2	3	4	5	6	7	8	9	10	11
number	1	1	5	1	10	1	100	16	36	1

tab:indecomposable

Question 3.15. Is it true that

$$\lim_{n \rightarrow \infty} \frac{\# \text{ decomposable solutions of size } n}{\# \text{ solutions of size } n} = 1?$$

The number of known indecomposable solutions is quite small for making reasonable conjectures. However, the following questions seem to be interesting.

Question 3.16. Let (X, r) be a finite solution. Is it true that if some σ_x contains a cycle of length coprime with $|X|$ then (X, r) is decomposable?

Question 3.17. Let (X, r) be a finite solution. How many fixed points of the diagonal of (X, r) guarantee the decomposability of (X, r) ?

Computer calculations suggest that if the diagonal of a solution (X, r) fixes more than $|X|/2$ points, then (X, r) is decomposable. We do not know, for example, what happens if the diagonal is a transposition or a 3-cycle.

Question 3.18. Are there indecomposable solutions, where the diagonal is a transposition, different from that of Example 3.14?

Question 3.19. Are there indecomposable solutions where the diagonal is a 3-cycle?

In Table 3.2 we show the cycle structure of the diagonal of small indecomposable solutions.

TABLE 3.2. Cycle structure of the diagonal of small indecomposable solutions.

n	number	cycle structure
4	3	4
	2	2
6	6	6
	3	3,3
	1	2,2,2
8	7	2,2
	25	2,2,2,2
	2	2,2,4
	1	2,6
	14	8
	51	4,4
9	1	3,3
	6	3,3,3
	9	9
10	1	2,2,2,2,2
	15	5,5
	20	10

tab:T

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