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On the cohomology of comodules over smash coproducts

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We consider the category of comodules over a smash coproduct coalgebra $C \bowtie H$. We show that there is a Grothendieck spectral sequence connecting the derived functors of the Hom functors coming from $C \bowtie H$ -colinear, H -colinear and rational C -colinear morphisms. We give several applications and connect our results to existing spectral sequences in the literature.

Keywords: Comodule coalgebra; Hopf algebra; smash coproduct; Hopf comodule; spectral sequence.

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Introduction

The category of comodules over a coalgebra over a field is a Grothendieck category with enough injectives, so we can consider right derived functors of left exact functors from the category of comodules to vector spaces. Examples of such functors include colinear homomorphisms, rational homomorphisms and coinvariants. The aim of this paper is to establish spectral sequences relating these derived functors. The classical procedure to do this is due to Grothendieck [6], and allows to associate a spectral sequence to a composition $G \circ F$ of two left exact functors, under the assumption that F preserves injective objects. This technique has been applied in the literature in several situations: in [11], Magid considers an affine algebraic group H acting on a commutative noetherian algebra R over an algebraically closed field, and studies the cohomology of $R \cdot H$ -modules; in [7] the cohomology of modules over a smash product algebra is studied; in [2], algebras with a Hopf algebra coaction

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are considered, and then the cohomology of the associated relative Hopf modules is investigated.

The setting of the present paper is as follows: take a Hopf algebra H over a field, and an H -comodule coalgebra C , that is a coalgebra in the category of (right) C -comodules. Then we can consider the smash coproduct coalgebra $C \bowtie H$, and the right derived functors of $\text{Hom}^{C \bowtie H}$. This functor is isomorphic to a composition of two functors of type Hom^H and HOM^C (Proposition 1.2); the second functor preserves injectives (Lemma 2.2), so that we can apply Grothendieck's results, leading to our main result Theorem 2.1.

Section 1 contains some preliminary results, mainly on comodules over the smash coproduct. Section 2 begins with properties of injective comodules, needed in order to prove the main result Theorem 2.1. We then look at particular situations; for example, the coinvariants functor $(-)^{\text{co}H}$ is isomorphic to $\text{Hom}^H(k, -)$, leading to a spectral sequence involving the coinvariants functor, see Corollary 2.1. In the case where H is cosemisimple, the exact sequence collapses, leading to Corollary 2.2. In Section 3, we present some applications. We present a new completely different proof of a result in [1] on the cosemisimplicity of the smash coproduct. As another consequence, we obtain the Hochschild-Serre spectral sequence for smash coproducts of coalgebras and Hopf algebras, we refer to [10], where the sequence has been established for groups extensions, to [3, Chap. XVI, Sec. 6], to [9,4] for rational algebraic group actions, and to [8] for smash products of algebras and groups. Some other results on homological properties in the category of comodules over smash coproducts can be found in [5,16,17].

Our results can be applied to the following smash coproducts, see Section 3 for detail:

- (1) $B \bowtie H$, where H is cosemisimple and $B = C^{\text{co}H}$.
- (2) $C \bowtie kG$, where G is a group and C is a G -graded coalgebra. Then the category of $C \bowtie kG$ -comodules is isomorphic to the category of G -graded C -comodules.
- (3) $A \bowtie k^G$, where A is a Hopf algebra, G is a finite group acting as Hopf algebra automorphisms on A and k^G is the algebra of set functions from G to k , see [14]. Note that k^G is a finite dimensional Hopf algebra.
- (4) $A(G) \cong A(L) \bowtie A(K)$, where G is an affine algebraic group, semi-direct product of algebraic subgroups L and K , with coordinate rings respectively denoted $A(G)$, $A(L)$ and $A(K)$, see [12]. Note that all these rings are commutative Hopf algebras, the rational G -modules and the $A(G)$ -comodules coincide, and $A(L)$ is an $A(K)$ -comodule coalgebra.

1. Preliminary results

1.1. Comodules over a Hopf algebra

Troughout this paper, we work over a field k . All unlabelled \otimes and Hom are over k . Let H be a Hopf algebra over k . The category \mathcal{M}^H consisting of right

H -comodules and right H -colinear maps is monoidal. The tensor product of two right H -comodules (M, ρ^M) and (N, ρ^N) is the usual tensor product $M \otimes N$, with coaction

$$\rho^{M \otimes N}(m \otimes n) = m_{[0]} \otimes n_{[0]} \otimes m_{[1]} n_{[1]}. \quad (1.1)$$

Here we use the Sweedler notation $\rho(m) = m_{[0]} \otimes m_{[1]}$ for the right H -coaction ρ on M . The subspace of H -coinvariants of $(M, \rho) \in \mathcal{M}^H$ is defined as

$$M^{\text{co}H} = \{m \in M \mid \rho^H(m) = m \otimes 1\} \cong \text{Hom}^H(k, M).$$

It is known that H is cosemisimple if and only if the coinvariants functor $(-)^{\text{co}H} : \mathcal{M}^H \rightarrow \mathcal{M}$ is exact.

For $M, N \in \mathcal{M}^H$, the canonical map

$$\iota : \text{Hom}(M, N) \otimes H \rightarrow \text{Hom}(M, N \otimes H), \quad \iota(f \otimes h)(m) = f(m) \otimes h$$

is injective (we work over a field), and is considered as an inclusion. $f \in \text{Hom}(M, N)$ is called H -rational [15] if $\rho : \text{Hom}(M, N) \rightarrow \text{Hom}(M, N \otimes H)$,

$$\rho(f)(m) = f(m_{[0]})_{[0]} \otimes f(m_{[0]})_{[1]} S(m_{[1]}),$$

factors through $\text{Hom}(M, N) \otimes H$. In this situation, we write $\rho(f) = f_{[0]} \otimes f_{[1]} \in \text{Hom}(M, N) \otimes H$. $\rho(f)$ is then characterized by the property

$$f_{[0]}(m) \otimes f_{[1]} = f(m_{[0]})_{[0]} \otimes f(m_{[0]})_{[1]} S(m_{[1]}), \quad (1.2)$$

or, equivalently,

$$\rho(f(m)) = f_{[0]}(m_{[0]}) \otimes f_{[1]} m_{[1]}, \quad (1.3)$$

for all $m \in M$. $\text{HOM}(M, N)$, the subspace of $\text{Hom}(M, N)$ consisting of rational maps is a right H -comodule. If M or H is finite dimensional, then the canonical inclusion ι is an isomorphism, and $\text{HOM}(M, N) = \text{Hom}(M, N)$. It follows from (1.3) that the coaction on $\text{HOM}(M, N)$ is designed in such a way that the evaluation map $\text{ev} : \text{HOM}(M, N) \otimes M \rightarrow N$ is right H -colinear. We remark that a different definition of rationality is used in [2]; actually the characterizing property (1.2) reduces to the one in [2] if H is replaced by H^{op} .

1.2. Comodules over a smash coproduct

A coalgebra in the category \mathcal{M}^H is called a right H -comodule coalgebra. This is a coalgebra C with a right H -coaction

$$\rho : C \rightarrow C \otimes H, \quad \rho(c) = c_{[0]} \otimes c_{[1]}$$

such that Δ_C and ε_C are right H -colinear, that is,

$$\Delta_C(c_{[0]}) \otimes c_{[1]} = c_{1[0]} \otimes c_{2[0]} \otimes c_{1[1]} c_{2[1]} \quad \text{and} \quad \varepsilon_C(c_{[0]}) c_{[1]} = \varepsilon_C(c) 1_H, \quad (1.4)$$

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for all $c \in C$. The smash coproduct $C \bowtie H$ is the vector space $C \otimes H$ with counit $\varepsilon_C \bowtie \varepsilon_H$ and comultiplication given by the formula

$$\Delta(c \bowtie h) = (c_1 \bowtie c_{2[1]}h_2) \otimes (c_{2[0]} \bowtie h_1), \quad (1.5)$$

for all $c \in C$ and $h \in H$. A direct verification shows that $C \bowtie H$ is a coalgebra, so we can consider the category of right $C \bowtie H$ -comodules $\mathcal{M}^{C \bowtie H}$. $\mathcal{M}^{C \bowtie H}$ has direct sums, and, since we are working over a field, it is a Grothendieck category. We can provide the following alternative description of $\mathcal{M}^{C \bowtie H}$. Recall from [1] that a right (C, H) -comodule is a vector space M together with a right C -coaction ρ^C and a right H -coaction ρ^H , with corresponding Sweedler notation

$$\rho^C(m) = m_{\{0\}} \otimes m_{\{1\}} \in M \otimes C \quad \text{and} \quad \rho^H(m) = m_{[0]} \otimes m_{[1]} \in M \otimes H,$$

satisfying the compatibility relation

$$\rho^C(m_{[0]}) \otimes m_{[1]} = m_{\{0\}[0]} \otimes m_{\{1\}[0]} \otimes m_{\{0\}[1]}m_{\{1\}[1]}, \quad (1.6)$$

for all $m \in M$. The category $\mathcal{M}^{C, H}$ of right (C, H) -comodules and right C -colinear and H -colinear maps is isomorphic to $\mathcal{M}^{C \bowtie H}$, see [1, Prop. 1.3]. We will now present some properties of $\mathcal{M}^{C \bowtie H}$.

Proposition 1.1. *Let C be a right H -comodule coalgebra, and take $M, N \in \mathcal{M}^{C, H}$. $\text{HOM}^C(M, N) = \text{Hom}^C(M, N) \cap \text{HOM}(M, N)$, the subspace of $\text{HOM}(M, N)$ consisting of right C -colinear maps, is an H -subcomodule of $\text{HOM}(M, N)$. This establishes a functor $\text{HOM}^C(M, -) : \mathcal{M}^{C, H} \rightarrow \mathcal{M}^H$. Furthermore*

$$\text{Hom}^{C \bowtie H}(M, N) = \text{HOM}^C(M, N)^{\text{co}H}.$$

Proof. Take $f \in \text{HOM}^C(M, N)$. Take a basis $\{h_i \mid i \in I\}$ of H as a vector space. Then we can write

$$\rho(f) = f_{[0]} \otimes f_{[1]} = \sum_i f_i \otimes h_i,$$

where only a finite number of the f_i are different from 0.

For a fixed $j \in I$, we can write

$$\Delta(h_j) = \sum_{i \in I} a_{ji} \otimes h_i, \quad (1.7)$$

for some $a_{ji} \in H$, with only finitely many of the a_{ji} different from 0.

For all $m \in M$, we have

$$\begin{aligned} & \rho^C(f_{[0]}(m)) \otimes f_{[1]} \stackrel{(1.2)}{=} \rho^C(f(m_{[0]})_{[0]}) \otimes f(m_{[0]})_{[1]}S(m_{[1]}) \\ & \stackrel{(1.6)}{=} f(m_{[0]})_{\{0\}[0]} \otimes f(m_{[0]})_{\{1\}[0]} \otimes f(m_{[0]})_{\{0\}[1]}f(m_{[0]})_{\{1\}[1]}S(m_{[1]}) \\ & \stackrel{(*)}{=} f(m_{[0]})_{\{0\}})_{[0]} \otimes m_{[0]})_{\{1\}[0]} \otimes f(m_{[0]})_{\{0\}})_{[1]}m_{[0]})_{\{1\}[1]}S(m_{[1]}) \\ & \stackrel{(1.6)}{=} f(m_{\{0\}[0]})_{[0]} \otimes m_{\{1\}[0]} \otimes f(m_{\{0\}[0]})_{[1]}m_{\{1\}[1]}S(m_{\{1\}[2]})S(m_{\{0\}[1]}) \\ & = f(m_{\{0\}[0]})_{[0]} \otimes m_{\{1\}} \otimes f(m_{\{0\}[0]})_{[1]}S(m_{\{0\}[1]}) \\ & \stackrel{(1.2)}{=} f_{[0]}(m_{\{0\}}) \otimes m_{\{1\}} \otimes f_{[1]}. \end{aligned}$$

At (*), we used the fact that f is right C -colinear. This can be rewritten as

$$\sum_i \rho^C(f_i(m)) \otimes h_i = \sum_i f_i(m_{\{0\}}) \otimes m_{\{1\}} \otimes h_i \in \bigoplus_{i \in I} N \otimes H \otimes kh_i. \quad (1.8)$$

For $l \in I$, take the projection of (1.8) onto $N \otimes H \otimes kh_l$. This gives

$$\rho^C(f_l(m)) = f_l(m_{\{0\}}) \otimes m_{\{1\}},$$

and therefore each f_l is right C -colinear.

A straightforward (and well-known) computation shows that

$$\rho(f_{[0]})(m) \otimes f_{[1]} = f_{[0]}(m) \otimes \Delta(f_{[1]}).$$

This can be rewritten as

$$\sum_{i \in I} \rho(f_i)(m) \otimes h_i = \sum_{j \in I} f_j(m) \otimes \Delta(h_j) \stackrel{(1.7)}{=} \sum_{i \in I} \sum_{j \in I} f_j(m) \otimes a_{ji} \otimes h_i. \quad (1.9)$$

For $l \in I$, take the projection of (1.9) onto $N \otimes H \otimes kh_l$. This gives

$$\rho(f_l)(m) = \sum_{j \in I} f_j(m) \otimes a_{jl}.$$

It follows that each f_l is rational. We now have shown that $f_l \in \text{HOM}^C(M, N)$, for all $l \in I$, and

$$\rho(f) = \sum_{l \in I} f_l \otimes h_l \in \text{HOM}^C(M, N) \otimes H.$$

We leave it to the reader to show that this construction is functorial. It is easy to show that $f \in \text{Hom}^H(M, N)$ if and only if $\rho(f) = f \otimes 1$, or, equivalently, $f \in \text{HOM}(M, N)^{\text{co}H}$. Consequently $f \in \text{Hom}^{C \rtimes H}(M, N)$ if and only if $f \in \text{Hom}^C(M, N) \cap \text{HOM}(M, N)^{\text{co}H} = \text{HOM}^C(M, N)^{\text{co}H}$. \square

Lemma 1.1. *Let M, N, P be $C \rtimes H$ -comodules. If $f \in \text{HOM}^C(M, N)$ and $g \in \text{HOM}^C(N, P)$, then $g \circ f \in \text{HOM}^C(M, P)$.*

Proof. It is well-known that $g \circ f$ is rational and C -colinear. \square

Lemma 1.2. *We have a functor $- \otimes C : \mathcal{M}^{C \rtimes H} \rightarrow \mathcal{M}^{C \rtimes H}$.*

Proof. Let M be a right (C, H) -comodule. C coacts on $M \otimes C$ as follows

$$\rho^C(m \otimes c) = m \otimes c_1 \otimes c_2. \quad (1.10)$$

Let us verify that the compatibility condition (1.6) holds.

$$\begin{aligned} \rho^C((m \otimes c)_{[0]}) \otimes (m \otimes c)_{[1]} &= m_{[0]} \otimes c_{[0]1} \otimes c_{[0]2} \otimes m_{[1]} c_{[1]} \\ &\stackrel{(1.4)}{=} m_{[0]} \otimes c_{1[0]} \otimes c_{2[0]} \otimes m_{[1]} c_{1[1]} c_{2[1]} \\ &= (m \otimes c_1)_{[0]} \otimes c_{2[0]} \otimes (m \otimes c_1)_{[1]} c_{2[1]} \\ &= (m \otimes c)_{\{0\}[0]} \otimes (m \otimes c)_{\{1\}[0]} \otimes (m \otimes c)_{\{0\}[1]} (m \otimes c)_{\{1\}[1]}. \end{aligned}$$

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We leave it to the reader to show that this construction is functorial. \square

Lemma 1.3. *Let $L \in \mathcal{M}^H$ and $M \in \mathcal{M}^{C \rtimes H}$. Then $L \otimes M$ is a right $C \rtimes H$ -comodule.*

Proof. C coacts on $L \otimes M$ as follows

$$\rho^C(l \otimes m) = (l \otimes m)_{\{0\}} \otimes (l \otimes m)_{\{1\}} = l \otimes m_{\{0\}} \otimes m_{\{1\}}. \quad (1.11)$$

Let us verify that the compatibility condition (1.6) holds.

$$\begin{aligned} \rho^C((l \otimes m)_{[0]}) \otimes (l \otimes m)_{[1]} &= l_{[0]} \otimes m_{[0]\{0\}} \otimes m_{[0]\{1\}} \otimes l_{[1]} m_{[1]} \\ &\stackrel{(1.6)}{=} l_{[0]} \otimes m_{\{0\}[0]} \otimes m_{\{1\}[0]} \otimes l_{[1]} m_{\{0\}[1]} m_{\{1\}[1]} \\ &= (l \otimes m_{\{0\}})_{[0]} \otimes m_{\{1\}[0]} \otimes (l \otimes m_{\{0\}})_{[1]} m_{\{1\}[1]} \\ &= (l \otimes m)_{\{0\}[0]} \otimes (l \otimes m)_{\{1\}[0]} \otimes (l \otimes m)_{\{0\}[1]} (l \otimes m)_{\{1\}[1]}. \quad \square \end{aligned}$$

Proposition 1.2. *Let $M, N \in \mathcal{M}^{C \rtimes H}$ and $L \in \mathcal{M}^H$. Then we have an isomorphism*

$$\mathrm{Hom}^{C \rtimes H}(L \otimes M, N) \cong \mathrm{Hom}^H(L, \mathrm{HOM}^C(M, N)).$$

Proof. We will first show that the canonical isomorphism

$$\phi : \mathrm{Hom}(L \otimes M, N) \rightarrow \mathrm{Hom}(L, \mathrm{Hom}(M, N)), \quad \phi(f)(l)(m) = f(l \otimes m)$$

restricts and corestricts to a map

$$\phi : \mathrm{Hom}^{C \rtimes H}(L \otimes M, N) \rightarrow \mathrm{Hom}^H(L, \mathrm{HOM}^C(M, N)).$$

Take $f \in \mathrm{Hom}^{C \rtimes H}(L \otimes M, N)$. For all $l \in L$ and $m \in M$, we have that

$$\begin{aligned} \rho(\phi(f)(l))(m) &= (\phi(f)(l)(m_{[0]}))_{[0]} \otimes (\phi(f)(l)(m_{[0]}))_{[1]} S(m_{[1]}) \\ &= f(l \otimes m_{[0]})_{[0]} \otimes f(l \otimes m_{[0]})_{[1]} S(m_{[1]}) \\ &\stackrel{(*)}{=} f((l \otimes m_{[0]})_{[0]}) \otimes (l \otimes m_{[0]})_{[1]} S(m_{[1]}) \\ &= f(l_{[0]} \otimes m_{[0]}) \otimes l_{[1]} m_{[1]} S(m_{[2]}) \\ &= f(l_{[0]} \otimes m) \otimes l_{[1]} = \phi(f)(l_{[0]})(m) \otimes l_{[1]}. \end{aligned}$$

At (*), we used the fact that f is H -colinear. This proves that $\phi(f)(l)$ is rational, and, moreover, that

$$\rho(\phi(f)(l)) = \phi(f)(l_{[0]}) \otimes l_{[1]}.$$

This tells us that $\phi(f) : L \rightarrow \mathrm{HOM}(M, N)$ is right H -colinear. It remains to be shown that $\phi(f)(l) : M \rightarrow N$ is right C -colinear, for every $l \in L$. Indeed, for all $m \in M$, we have that

$$\rho^C((\phi(f)(l))(m)) = \rho^C(f(l \otimes m)) \stackrel{(*)}{=} f(l \otimes m_{\{0\}}) \otimes m_{\{1\}} = (\phi(f)(l))(m_{\{0\}}) \otimes m_{\{1\}}.$$

At (*), we used the fact that f is right C -colinear. The inverse $\phi^{-1} : \text{Hom}(L, \text{Hom}(M, N)) \rightarrow \text{Hom}(L \otimes M, N)$ of ϕ is given by the formula

$$\phi^{-1}(g)(l \otimes m) = (g(l))(m).$$

The proof is finished if we can show that ϕ^{-1} restricts and corestricts to a map $\phi^{-1} : \text{Hom}^H(L, \text{HOM}^C(M, N)) \rightarrow \text{Hom}^{C \rtimes H}(L \otimes M, N)$. Take a right H -colinear map $g : L \rightarrow \text{HOM}^C(M, N)$. For all $l \in L$ and $m \in M$, we have that

$$\begin{aligned} \rho^C(\phi^{-1}(g)(l \otimes m)) &= \rho^C((g(l))(m)) \stackrel{(*)}{=} g(l)(m_{\{0\}}) \otimes m_{\{1\}} \\ &= \phi^{-1}(g)(l \otimes m_{\{0\}}) \otimes m_{\{1\}} = \phi^{-1}(g)((l \otimes m)_{\{0\}}) \otimes (l \otimes m)_{\{1\}}, \end{aligned}$$

and it follows that $\phi^{-1}(g)$ is right C -colinear. As $g : L \rightarrow \text{HOM}(M, N)$ is right H -linear, we have that $\rho(g(l)) = g(l_{[0]}) \otimes l_{[1]}$, and

$$g(l)_{[0]}(m) \otimes g(l)_{[1]} = g(l_{[0]})(m) \otimes l_{[1]}, \quad (1.12)$$

for all $m \in M$. Now

$$\begin{aligned} \phi^{-1}(g)((l \otimes m)_{[0]}) \otimes (l \otimes m)_{[1]} &= \phi^{-1}(g)(l_{[0]} \otimes m_{[0]}) \otimes l_{[1]}m_{[1]} \\ &= (g(l_{[0]}))(m_{[0]}) \otimes l_{[1]}m_{[1]} \stackrel{(1.12)}{=} g(l)_{[0]}(m_{[0]})g(l)_{[1]}m_{[1]} \\ &\stackrel{(1.2)}{=} (g(l))(m_{[0]})_{[0]} \otimes (g(l))(m_{[0]})_{[1]}S(m_{[1]})m_{[2]} \\ &= \rho^H((g(l))(m)) = \rho^H(\phi^{-1}(g)(l \otimes m)), \end{aligned}$$

and this proves that $\phi^{-1}(g)$ is right H -colinear. \square

Lemma 1.4. *We make the following additional assumptions: H is commutative and C is cocommutative. Furthermore M is finite dimensional, or H and C are finite dimensional. If $M, N \in \mathcal{M}^{C \rtimes H}$, then $\text{Hom}^C(M, N) \in \mathcal{M}^{C \rtimes H}$.*

Proof. The H -coaction is defined as in Proposition 1.1, taking into account that $\text{Hom}^C(M, N) = \text{HOM}^C(M, N)$. The C -coaction is defined as follows: for $f \in \text{Hom}^C(M, N)$, $\rho^C(f) = f_{\{0\}} \otimes f_{\{1\}} \in \text{Hom}(M, N \otimes C) \cong \text{Hom}(M, N) \otimes C$ if and only if

$$f_{\{0\}}(m) \otimes f_{\{1\}} = f(m_{\{0\}}) \otimes m_{\{1\}}, \quad (1.13)$$

for all $m \in M$. Fix a basis $\{c_i \mid i \in I\}$ of C . There exist $f_i \in \text{Hom}(M, N)$ such that $\rho^C(f) = \sum_i f_i \otimes c_i$. For all $m \in M$, we have that

$$\begin{aligned} \sum_i \rho^C(f_i(m)) \otimes c_i &= f_{\{0\}}(m)_{\{0\}} \otimes f_{\{0\}}(m)_{\{1\}} \otimes f_{\{1\}} \\ &\stackrel{(1.13)}{=} f(m_{\{0\}})_{\{0\}} \otimes f((m_{\{0\}})_{\{1\}} \otimes m_{\{1\}}) \stackrel{(*)}{=} f(m_{\{0\}}) \otimes m_{\{1\}} \otimes m_{\{2\}} \\ &\stackrel{(**)}{=} f_{\{0\}}(m_{\{0\}}) \otimes m_{\{1\}} \otimes f_{\{1\}} = \sum_i f_i(m_{\{0\}}) \otimes m_{\{1\}} \otimes c_i. \end{aligned}$$

At (*), we used the C -colinearity of f ; at (**), we used the cocommutativity of C . It follows that $\rho^C(f_i(m)) = f_i(m_{\{0\}}) \otimes m_{\{1\}}$, for all $m \in M$ and $i \in I$. We conclude

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that each f_i is right C -colinear, and $\rho^C(f) \in \text{Hom}^C(M, N) \otimes C$. Let us show that ρ^C is coassociative. For all $m \in M$, we have that

$$\begin{aligned} f_{\{0\}\{0\}}(m) \otimes f_{\{0\}\{1\}} \otimes f_{\{1\}} &\stackrel{(1.13)}{=} f_{\{0\}}(m_{\{0\}}) \otimes m_{\{1\}} \otimes f_{\{1\}} \\ &\stackrel{(1.13)}{=} f_{\{0\}}(m_{\{0\}}) \otimes m_{\{2\}} \otimes m_{\{1\}} \stackrel{(*)}{=} f(m_{\{0\}}) \otimes \Delta_C(m_{\{1\}}) \\ &\stackrel{(1.13)}{=} f_{\{0\}}(m) \otimes \Delta_C(f_{\{1\}}). \end{aligned}$$

At (*), we used the cocommutativity of C . It follows that $\rho^C(f_{\{0\}}) \otimes f_{\{1\}} = f_{\{0\}} \otimes \Delta_C(f_{\{1\}})$, so that ρ^C is coassociative. It is obvious that

$$\varepsilon_C(f_{\{1\}})f_{\{0\}}(m) = f(m_{\{0\}})\varepsilon_C(m_{\{1\}}) = f(m),$$

and the counit property follows. Finally, we have to prove the compatibility condition (1.6). For all $m \in M$, we have that

$$\begin{aligned} f_{\{0\}[0]}(m) \otimes f_{\{1}[0]} \otimes f_{\{0\}[1]}f_{\{1\}[1]} &\stackrel{(1.2)}{=} f_{\{0\}}(m_{\{0\}})_{[0]} \otimes f_{\{1\}[0]} \otimes f_{\{0\}}(m_{\{0\}})_{[1]}S(m_{\{1\}})f_{\{1\}[1]} \\ &\stackrel{(1.13)}{=} f(m_{\{0\}\{0\}})_{[0]} \otimes m_{\{0\}\{1\}[0]} \otimes f(m_{\{0\}\{0\}})_{[1]}S(m_{\{1\}})m_{\{0\}\{1\}[1]} \\ &\stackrel{(1.6)}{=} f(m_{\{0\}[0]})_{[0]} \otimes m_{\{1\}[0]} \otimes f(m_{\{0\}[0]})_{[1]}S(m_{\{1\}[2]})S(m_{\{0\}[1]})m_{\{1\}[1]} \\ &\stackrel{(*)}{=} f(m_{\{0\}[0]})_{[0]} \otimes m_{\{1\}} \otimes f(m_{\{0\}[0]})_{[1]}S(m_{\{0\}[1]}) \\ &\stackrel{(1.2)}{=} f_{[0]}(m_{\{0\}}) \otimes m_{\{1\}} \otimes f_{[1]} = f_{[0]\{0\}}(m) \otimes f_{[0]\{1\}} \otimes f_{[1]}, \end{aligned}$$

and the compatibility relation follows; At (*), we used the fact that H is commutative. \square

2. A spectral sequence

Lemma 2.1. *Let I be a right (C, H) -comodule. We have two left exact functors (resp. covariant and contravariant) $\text{HOM}^C(I, -)$, $\text{HOM}^C(-, I) : \mathcal{M}^{C \rtimes H} \rightarrow \mathcal{M}^H$.*

Proof. In Proposition 1.1 the two functors are defined at the level of objects. It is easy to see that this construction is functorial, and that the two functors are left exact. \square

It is well-known that the category of comodules over a coalgebra over a field is a Grothendieck category. Therefore the categories \mathcal{M}^C , \mathcal{M}^H and $\mathcal{M}^{C \rtimes H}$ are Grothendieck categories. For a right (C, H) -comodule M , we will establish a spectral sequence connecting the right derived functors of $\text{HOM}^C(M, -)$, $\text{Hom}^{C \rtimes H}(M, -)$ and $(-)^{\text{co}H}$.

Lemma 2.2. *Take right (C, H) -comodules M and I , and assume that I is injective in $\mathcal{M}^{C \rtimes H}$. Then $\text{HOM}^C(M, I)$ is an injective object in \mathcal{M}^H .*

Proof. Consider an exact sequence $0 \rightarrow L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow 0$ in \mathcal{M}^H , and, a fortiori, in \mathcal{M}_k . The sequence $0 \rightarrow L_1 \otimes M \rightarrow L_2 \otimes M \rightarrow L_3 \otimes M \rightarrow 0$ is exact in

\mathcal{M}_k ; the spaces and maps in the sequence are in $\mathcal{M}^{C \rtimes H}$, hence we have an exact sequence in $\mathcal{M}^{C \rtimes H}$. Now $I \in \mathcal{M}^{C \rtimes H}$, so

$$\begin{aligned} 0 &\rightarrow \text{Hom}^{C \rtimes H}(L_3 \otimes M, I) \rightarrow \text{Hom}^{C \rtimes H}(L_2 \otimes M, I) \\ &\rightarrow \text{Hom}^{C \rtimes H}(L_1 \otimes M, I) \rightarrow 0 \end{aligned}$$

is an exact sequence in \mathcal{M}_k . It follows from Proposition 1.2 that this sequence is isomorphic to

$$\begin{aligned} 0 &\rightarrow \text{Hom}^H(L_3, \text{HOM}^C(M, I)) \rightarrow \text{Hom}^H(L_2, \text{HOM}^C(M, I)) \\ &\rightarrow \text{Hom}^H(L_1, \text{HOM}^C(M, I)) \rightarrow 0. \end{aligned}$$

This shows that the functor $\text{Hom}^H(-, \text{HOM}^C(M, I)) : \mathcal{M}^H \rightarrow \mathcal{M}_k$ is exact, proving the assertion. \square

Lemma 2.3. *If $I \in \mathcal{M}^{C \rtimes H}$ is injective, then $\text{HOM}^C(-, I) : \mathcal{M}^{C \rtimes H} \rightarrow \mathcal{M}^H$ is an exact functor.*

Proof. We know from Lemma 2.1 that $\text{HOM}^C(-, I)$ is left exact. Let $i : M \rightarrow N$ be a monomorphism in $\mathcal{M}^{C \rtimes H}$. We need to show that $\text{HOM}^C(i, I) : \text{HOM}^C(N, I) \rightarrow \text{HOM}^C(M, I)$ is surjective.

Take $f \in \text{HOM}^C(M, I)$, and let V be a finite dimensional H -subcomodule of $\text{HOM}^C(M, I)$ containing f . By Lemma 1.3, $V \otimes M, V \otimes N \in \mathcal{M}^{C \rtimes H}$. The map $V \otimes i : V \otimes M \rightarrow V \otimes N$ is injective, and $C \rtimes H$ -colinear (see the $C \rtimes H$ -coaction on $V \otimes M$ and $V \otimes N$ as described in the proof of Lemma 1.3). This shows that $V \otimes i$ is a monomorphism in $\mathcal{M}^{C \rtimes H}$. We have seen that the evaluation map $\text{ev} : V \otimes M \rightarrow I$, $F(v \otimes m) = v(m)$ is right H -colinear. It follows from Lemma 1.3 that $V \otimes M \in \mathcal{M}^{C \rtimes H}$. Now we claim that ev is also right C -colinear. Indeed, for all $v \in V$ and $m \in M$, we have that

$$\rho^C(\text{ev}(v \otimes m)) = \rho^C(v(m)) \stackrel{(*)}{=} v(m_{\{0\}}) \otimes m_{\{1\}} = (\text{ev} \otimes C)(\rho^C(v \otimes m)).$$

At (*), we used the right C -colinearity of v . It follows that ev is a morphism in $\mathcal{M}^{C \rtimes H}$, and the injectivity of $I \in \mathcal{M}^{C \rtimes H}$ entails the existence of a $C \rtimes H$ -colinear map $G : V \otimes N \rightarrow I$ such that $G \circ (V \otimes i) = \text{ev}$. Consider the map $g : N \rightarrow I$, $g(n) = G(f \otimes n)$. For all $n \in N$, we have that

$$\begin{aligned} g(n_{[0][0]} \otimes g(n_{[0][1]}S(n_{[1]})) &= G(f \otimes n_{[0][0]}) \otimes G(f \otimes n_{[0][1]}S(n_{[1]})) \\ &\stackrel{(*)}{=} G(f_{[0]} \otimes n_{[0]}) \otimes f_{[1]}n_{[1]}S(n_{[2]}) = G(f_{[0]} \otimes n) \otimes f_{[1]}. \end{aligned}$$

At (*), we used the right C -colinearity of G . This proves that g is H -rational, and $\rho^H(g) = G(f_{[0]} \otimes -) \otimes f_{[1]}$. Furthermore

$$\rho^C(g(n)) = \rho^C(G(f \otimes n)) \stackrel{(*)}{=} G(f \otimes n_{\{0\}}) \otimes n_{\{1\}} = g(n_{\{0\}}) \otimes n_{\{1\}},$$

proving that g is right C -colinear. At (*), we used the right C -colinearity of G . We conclude that $g \in \text{HOM}^C(N, I)$, and, finally $\text{HOM}^C(i, I)(g) = g \circ i = f$, since

$$f(m) = \text{ev}(f \otimes m) = (G \circ (V \otimes i))(f \otimes m) = (g \circ i)(m),$$

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for all $m \in M$. This completes the proof. \square

Lemma 2.4.

- (1) If $I \in \mathcal{M}^{C \rtimes H}$ is injective, then I is also injective as an object of \mathcal{M}^C .
- (2) Assume that H is cosemisimple, and that $M \in \mathcal{M}^{C \rtimes H}$ is finite dimensional and projective as a right C -comodule. Then M is also projective as a right $C \rtimes H$ -comodule.

Proof. (1) We have seen in Lemma 1.1 that $I \otimes C \in \mathcal{M}^{C \rtimes H}$. For all $u \in I$, we have

$$\rho^H(\rho^C(u)) = u_{\{0\}[0]} \otimes u_{\{1\}[0]} \otimes u_{\{0\}[1]} u_{\{1\}[1]} \stackrel{(1.6)}{=} \rho^C(u_{[0]}) \otimes u_{[1]},$$

so that $\rho^C : I \rightarrow I \otimes C$ is right H -colinear. It is well-known that ρ^C is a monomorphism in \mathcal{M}^C , and we can conclude that ρ^C is a monomorphism in $\mathcal{M}^{C \rtimes H}$. Thus we have a short exact sequence

$$0 \rightarrow I \xrightarrow{\rho^C} I \otimes C \rightarrow (I \otimes C)/I \rightarrow 0$$

in $\mathcal{M}^{C \rtimes H}$. Since $I \in \mathcal{M}^{C \rtimes H}$ is injective, this sequence splits in $\mathcal{M}^{C \rtimes H}$, and, a fortiori, in \mathcal{M}^C . Hence I is a direct factor of $I \otimes C$ in \mathcal{M}^C . It is well-known that $I \otimes C$ is injective as a C -comodule, and we conclude that I is injective as a C -comodule.

(2) Take an exact sequence

$$0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$$

in $\mathcal{M}^{C \rtimes H}$, and, a fortiori, in \mathcal{M}^C . $M \in \mathcal{M}^C$ is projective, so the sequence

$$0 \rightarrow \text{Hom}^C(M, N_1) \rightarrow \text{Hom}^C(M, N_2) \rightarrow \text{Hom}^C(M, N_3) \rightarrow 0$$

is exact in \mathcal{M}^C . The functor $(-)^{\text{co}H} : \mathcal{M}^C \rightarrow \mathcal{M}_k$ is exact since H is cosemisimple, so we have an exact sequence

$$0 \rightarrow \text{Hom}^C(M, N_1)^{\text{co}H} \rightarrow \text{Hom}^C(M, N_2)^{\text{co}H} \rightarrow \text{Hom}^C(M, N_3)^{\text{co}H} \rightarrow 0$$

in \mathcal{M}_k . By Proposition 1.1, this sequence can be rewritten as

$$0 \rightarrow \text{Hom}^{C \rtimes H}(M, N_1) \rightarrow \text{Hom}^{C \rtimes H}(M, N_2) \rightarrow \text{Hom}^{C \rtimes H}(M, N_3) \rightarrow 0.$$

We conclude that $\text{Hom}^{C \rtimes H}(M, -)$ is an exact functor, as needed. \square

For $M \in \mathcal{M}^C$, let $\text{Ext}^C(M, -)$ be the right derived functors of the functor $\text{Hom}^C(M, -) : \mathcal{M}^C \rightarrow \mathcal{M}_k$. For $M, N \in \mathcal{M}^C$, $\text{Ext}^{C,p}(M, N) = H^p(\text{Hom}^C(M, \mathcal{E}^*))$, the p -th cohomology group of the complex $\text{Hom}^C(M, \mathcal{E}^*)$ associated to an injective resolution \mathcal{E}^* of N in \mathcal{M}^C .

If M is finite dimensional, then $M^* \in {}^C\mathcal{M}$ with left C -coaction

$$\lambda(m^*) = \sum_i \langle m^*, m_{i[0]} \rangle m_{i[1]} \otimes m_i^*,$$

for all $m^* \in M^*$, where $\sum_i m_i \otimes m_i^* \in M \otimes M^*$ is the finite dual basis of M . The natural isomorphism $- \otimes M^* \cong \text{Hom}(M, -) : \mathcal{M}^C \rightarrow \mathcal{M}_k$ restricts to a natural isomorphism $-\square_C M^* \cong \text{Hom}^C(M, -)$, where \square_C is the cotensor product over C . This implies immediately that we have natural isomorphism $\text{Tor}^C(N, M^*) \cong \text{Ext}^C(M, N)$ between the right derived functors of $-\square_C M^*$ and $\text{Hom}^C(M, -)$. Take $M \in \mathcal{M}^{C \rtimes H}$, and let $\text{EXT}^C(M, -)$ be the right derived functors of the functor $\text{HOM}^C(M, -)$ from Proposition 1.1.

Lemma 2.5. *Take $M, N \in \mathcal{M}^{C \rtimes H}$, and assume that M or H is finite dimensional. Then $\text{EXT}^C(M, N) = \text{Ext}^C(M, N)$ as vector spaces.*

Proof. Take an injective resolution \mathcal{E}^* of $N \in \mathcal{M}^{C \rtimes H}$. It follows from Lemma 2.4 that \mathcal{E}^* is also an injective resolution of $N \in \mathcal{M}^C$. Since M or H is finite dimensional, we have that $\text{HOM}^C(M, P) = \text{Hom}^C(M, P)$ for all $P \in \mathcal{M}^{C \rtimes H}$, and consequently

$$\text{EXT}^C(M, N) = H^\bullet(\text{HOM}^C(M, \mathcal{E}^*)) = H^\bullet(\text{Hom}^C(M, \mathcal{E}^*)) = \text{Ext}^C(M, N). \quad \square$$

Theorem 2.1. *For $M, N \in \mathcal{M}^{C \rtimes H}$ and $L \in \mathcal{M}^H$, we have a spectral sequence*

$$\text{Ext}^{H,p}(L, \text{EXT}^{C,q}(M, N)) \Rightarrow \text{Ext}^{C \rtimes H, p+q}(L \otimes M, N), \quad (2.1)$$

for all $p, q \geq 0$.

Proof. We have left exact functors $F = \text{HOM}^C(M, -) : \mathcal{M}^{C \rtimes H} \rightarrow \mathcal{M}^H$ and $G = \text{Hom}^H(L, -) : \mathcal{M}^H \rightarrow \mathcal{M}_k$. Now $G \circ F = \text{Hom}^{C \rtimes H}(L \otimes M, N)$ (Proposition 1.2) and F preserves injectives (Lemma 2.2), so we have the Grothendieck spectral sequence for composite functors

$$(R^p G)((R^q F)(N)) \Rightarrow R^{p+q}(G \circ F)(N),$$

for $p, q \geq 0$, see [6, Theorem 2.4.1], which specifies to (2.1). \square

We have seen in Section 1 that the coinvariants functor $(-)^{\text{co}H}$ is naturally isomorphic to $\text{Hom}^H(k, -)$. This implies that the right derived functors $R^p(-)^{\text{co}H}$ are naturally isomorphic to $\text{Ext}^H(k, -)$, and we obtain the following corollary from Theorem 2.1.

Corollary 2.1. *For $M, N \in \mathcal{M}^{C \rtimes H}$, we have a spectral sequence*

$$R^p(\text{EXT}^{C,q}(M, N))^{\text{co}H} \Rightarrow \text{Ext}^{C \rtimes H, p+q}(M, N).$$

Corollary 2.2. *$M, N \in \mathcal{M}^{C \rtimes H}$ and $L \in \mathcal{M}^H$. If H is cosemisimple, then we have isomorphisms of vector spaces*

$$\text{Hom}^H(L, \text{EXT}^{C,q}(M, N)) \cong \text{Ext}^{C \rtimes H, q}(L \otimes M, N); \quad (2.2)$$

$$\text{EXT}^{C,q}(M, N)^{\text{co}H} \cong \text{Ext}^{C \rtimes H, q}(M, N), \quad (2.3)$$

for all $q \geq 0$.

Proof. The category \mathcal{M}^H is completely reducible if H is cosemisimple, and then the spectral sequence (2.1) collapses, yielding (2.2). (2.3) follows after we take $L = k$ in (2.2). \square

It follows from Lemma 2.5 that we can replace EXT^C by Ext^C in Theorem 2.1 and Corollaries 2.1 and 2.2, in the situation where M or H is finite dimensional.

Corollary 2.3. *Assume that H is cosemisimple and take $M \in \mathcal{M}^{C \rtimes H}$.*

- (1) *If M is projective as a C -comodule and M or H is finite dimensional, then $L \otimes M$ is projective in $\mathcal{M}^{C \rtimes H}$, for every $L \in \mathcal{M}^H$.*
- (2) *If H is finite dimensional, then M is injective in \mathcal{M}^C if and only if M is injective in $\mathcal{M}^{C \rtimes H}$.*

Proof. (1) Since M is projective in \mathcal{M}^C , $\text{EXT}^{C,q}(M, N) = \text{Ext}^{C,q}(M, N) = 0$ for all $q > 0$ and $N \in \mathcal{M}^{C \rtimes H}$. It then follows from (2.2) that $\text{Ext}^{C \rtimes H, q}(L \otimes M, N) \cong \text{Hom}^H(L, \text{EXT}^{C,q}(M, N)) = 0$ for all $q > 0$ and $N \in \mathcal{M}^{C \rtimes H}$, and this implies that $L \otimes M$ is projective as a right $C \rtimes H$ -comodule.

(2) If M is injective in \mathcal{M}^C , then $\text{Ext}^{C,1}(N, M) = 0$ for all $N \in \mathcal{M}^{C \rtimes H}$, and it follows from (2.2) (with $L = k$) that $\text{Ext}^{C \rtimes H, 1}(N, M) = 0$, for all $N \in \mathcal{M}^{C \rtimes H}$, and M is injective in $\mathcal{M}^{C \rtimes H}$. The converse implication follows from Lemma 2.4. \square

3. Further applications

3.1. Cosemisimplicity of the smash coproduct

Proposition 3.1 is well-known, see [1, Cor. 5.3]. We present a completely different proof.

Proposition 3.1. *If C and H are cosemisimple, then every $M \in \mathcal{M}^{C \rtimes H}$ is a direct sum in $\mathcal{M}^{C \rtimes H}$ of a family of simple $\mathcal{M}^{C \rtimes H}$ -subcomodules. Hence every object of $\mathcal{M}^{C \rtimes H}$ is semisimple, that is, $\mathcal{M}^{C \rtimes H}$ is a semisimple category.*

Proof. Assume first that M is finite dimensional. It follows from Lemma 2.4(2) or Corollary 2.3(2) (with $L = k$) that M is projective in $\mathcal{M}^{C \rtimes H}$. Let N be a $C \rtimes H$ -subcomodule N of M . $M/N \in \mathcal{M}^{C \rtimes H}$ is finite dimensional, and therefore projective, so that the short exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$$

splits in $\mathcal{M}^{C \rtimes H}$. This means that every $C \rtimes H$ -subcomodule of M is a direct summand, and M is the direct sum of a finite family of simple $C \rtimes H$ -subcomodules. Now assume that the dimension of M is possibly infinite. By the Fundamental Theorem for comodules, every $m \in M$ is contained in a finite dimensional $C \rtimes H$ -subcomodule V_m of M , see for example [13, 5.1.1] V_m is the direct sum of a finite number of simple $C \rtimes H$ -subcomodules of V_m and, a fortiori, of M . Thus every

$m \in M$ is contained in a direct sum of simple $C \rtimes H$ -subcomodules of M , and we can easily conclude that M is a direct sum of a family of simple $C \rtimes H$ -subcomodules. \square

3.2. The smash coproduct $B \rtimes H$

Assume that H is cosemisimple, and let ϕ be a left integral in H^* . Then we have the trace map

$$\Psi : C \rightarrow C^{\text{co}H}, \quad \Psi(c) = \phi(c_{[1]})c_{[0]}.$$

It is known that $B = C^{\text{co}H}$ is a coalgebra with comultiplication

$$\Delta'(c) = \Psi(c_1) \otimes c_2 = c_1 \otimes \Psi(c_2) = \Psi(c_1) \otimes \Psi(c_2),$$

for all $c \in C^{\text{co}H}$, and that Ψ is a coalgebra map, see [1, Cor. 2.4]. For a right $C \rtimes H$ -comodule M , we have a k -linear map

$$\Psi_M : M \rightarrow M^{\text{co}H}, \quad \Psi_M(m) = \phi(m_{[1]})m_{[0]}.$$

Observe that $\Psi_C = \Psi$. $M^{\text{co}H}$ is a right B -comodule, with coaction

$$\begin{aligned} \rho'(m) &= m_{(0)} \otimes m_{(1)} = \Psi_M(m)(m_{\{0\}}) \otimes m_{\{1\}} \\ &= m_{\{0\}} \otimes \Psi(m_{\{1\}}) = \Psi_M(m)(m_{\{0\}}) \otimes \Psi(m_{\{1\}}), \end{aligned}$$

for all $m \in M^{\text{co}H}$. A morphism $f : M \rightarrow N$ in $\mathcal{M}^{C \rtimes H}$ restricts and corestricts to a right B -colinear $f^{\text{co}H} : M^{\text{co}H} \rightarrow N^{\text{co}H}$, so that we have a functor $(-)^{\text{co}H} : \mathcal{M}^{C \rtimes H} \rightarrow \mathcal{M}^B$.

Now we consider B as a right H -comodule coalgebra, under trivial H -coaction $\rho(b) = b \otimes 1$, for all $b \in B$. Then we can consider the smash coproduct $B \rtimes H$, and our previous results remain valid if we replace C by B .

Theorem 3.1. *Assume that H is cosemisimple and take $M \in \mathcal{M}^B$ and $N \in \mathcal{M}^{B \rtimes H}$. M is viewed as a right $B \rtimes H$ -comodule under the trivial H -coaction. Then*

$$\text{Ext}^{B \rtimes H}(M, N)^{\text{co}H} = \text{Ext}^B(M, N^{\text{co}H}). \quad (3.1)$$

Proof. We will first show that

$$\text{Hom}^{B \rtimes H}(M, N) = \text{HOM}^B(M, N)^{\text{co}H} = \text{Hom}^B(M, N^{\text{co}H}). \quad (3.2)$$

The first equality follows from Proposition 1.1. Take $f \in \text{HOM}^B(M, N)^{\text{co}H}$. Then $\rho(f) = f \otimes 1$, and (1.2) takes the form $f(m) \otimes 1 = \rho(f(m))$, for all $m \in M$, hence $f(m) \in N^{\text{co}H}$, and it follows that $f \in \text{Hom}^B(M, N^{\text{co}H})$.

Conversely, let $f \in \text{Hom}^B(M, N^{\text{co}H})$, and view f as a map $M \rightarrow N$. For all $m \in M$, $\rho(f(m)) = f(m) \otimes 1$, and we deduce from (1.2) that f is H -rational and $\rho(f) = f \otimes 1$, hence $f \in \text{HOM}^B(M, N)^{\text{co}H}$. This proves the second equality.

Let I be an injective object in $\mathcal{M}^{B \rtimes H}$. In view of (2.2), $\text{EXT}^{B, q}(M, I)^{\text{co}H} \cong$

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$$\text{Ext}^{B \rtimes H, q}(M, I) = 0.$$

We claim that $I^{\text{co}H}$ is injective in \mathcal{M}^B . Take an exact sequence $0 \rightarrow M_3 \rightarrow M_2 \rightarrow M_1 \rightarrow 0$ in \mathcal{M}^B . Viewing the M_i as $B \rtimes H$ -comodules via the trivial H -coaction, this sequence is also exact in $\mathcal{M}^{B \rtimes H}$. From the injectivity of $I \in \mathcal{M}^{B \rtimes H}$, it follows that

$$0 \rightarrow \text{Hom}^{B \rtimes H}(M_1, I) \rightarrow \text{Hom}^{B \rtimes H}(M_2, I) \rightarrow \text{Hom}^{B \rtimes H}(M_3, I) \rightarrow 0$$

is exact in \mathcal{M}_k . In view of (3.2), this sequence is isomorphic to

$$0 \rightarrow \text{HOM}^B(M_1, I^{\text{co}H}) \rightarrow \text{HOM}^B(M_2, I^{\text{co}H}) \rightarrow \text{HOM}^B(M_3, I^{\text{co}H}) \rightarrow 0.$$

It follows that $\text{Ext}^{B, q}(M, I^{\text{co}H}) = 0$, for all $q > 0$. Observe that functors

$$\text{EXT}^B(M, -)^{\text{co}H}, \text{Ext}^B(M, (-)^{\text{co}H}) : \mathcal{M}^{B \rtimes H} \rightarrow \mathcal{M}_k.$$

are cohomological in the sense of [6, 2.2]. Indeed, right derived functors are cohomological, the coinvariants functor is exact since H is cosemisimple, and it is obvious that the composition of a cohomological functor and an exact functor is cohomological. The two functors coincide in degree 0 (3.2) and vanish in degree > 0 at injective objects of $\mathcal{M}^{B \rtimes H}$. Therefore $\text{Ext}^B(M, (-)^{\text{co}H}) \cong \text{EXT}^B(M, -)^{\text{co}H}$, which is isomorphic to $\text{Ext}^{B \rtimes H}(M, -)$ by (2.2). \square

3.3. Hochschild-Serre spectral sequence for smash coproducts

Assume that C contains a H -coinvariant grouplike element x . This is the case for the examples (3) and (4) mentioned in the introduction. If H is cosemisimple, then x is also a grouplike element of the coalgebra $B = C^{\text{co}H}$ discussed in the previous subsection. For a right C -comodule N , we can consider the subspace of C -coinvariant elements

$$N^{\text{co}C} = \{n \in N \mid n_{\{0\}} \otimes n_{\{1\}} = n \otimes x\}.$$

Lemma 3.1. *If $N \in \mathcal{M}^{C \rtimes H}$, then $N^{\text{co}C} \in \mathcal{M}^H$.*

Proof. Fix a basis $\{h_i \mid i \in I\}$ of H . Take $n \in N^{\text{co}C}$ and write $\rho^H(n) = n_{[0]} \otimes n_{[1]} = \sum_i n_i \otimes h_i$. Then

$$\begin{aligned} \sum_i \rho^C(n_i) \otimes h_i &= \rho^C(n_{[0]}) \otimes n_{[1]} \stackrel{(1.6)}{=} n_{\{0\}[0]} \otimes n_{\{1\}[0]} \otimes n_{\{0\}[1]} n_{\{1\}[1]} \\ &= n_{[0]} \otimes x_{[0]} \otimes n_{[1]} x_{[1]} = n_{[0]} \otimes x \otimes n_{[1]} = \sum_i n_i \otimes x \otimes h_i \end{aligned}$$

It follows that each n_i is C -coinvariant, and $\rho(n) \in N^{\text{co}C} \otimes H$. \square

Observe that $x \rtimes 1_H \in G(C \rtimes H)$. k is a right $C \rtimes H$ -comodule with coaction $\rho(\lambda) = \lambda x \rtimes 1_H$, and $N^{\text{co}C} \cong \text{Hom}^C(k, N)$. Consequently we have isomorphisms of derived functors $\text{Ext}^C(k, -) \cong R^p(-)^{\text{co}C}$. Taking $M = k$ in Corollary 2.1, we obtain the following spectral sequence.

Theorem 3.2. *Assume that C contains a H -coinvariant grouplike element x . For $N \in \mathcal{M}^{C \bowtie H}$, we have a spectral sequence*

$$R^p(R^q(N)^{\text{co}C})^{\text{co}H} \Rightarrow R^{p+q}(N)^{\text{co}C \bowtie H}.$$

Applying Theorem 3.2 to Examples (3) and (4) from the introduction, we obtain the following results.

Corollary 3.1. *Let A be a Hopf algebra, and let G be a finite group acting as a group of Hopf algebra automorphisms on A . For $N \in \mathcal{M}^{A \bowtie k^G}$, we have a spectral sequence*

$$R^p(R^q(N)^{\text{co}A})^{\text{co}k^G} \Rightarrow R^{p+q}(N)^{\text{co}A \bowtie k^G}.$$

Corollary 3.2. *Assume that an affine algebraic group G can be written as the semi-direct product of two algebraic subgroups L and K . Let $A(G)$ be the coordinate ring of G . For $N \in \mathcal{M}^{A(G)}$, we have a spectral sequence*

$$R^p(R^q(N)^{\text{co}A(L)})^{\text{co}A(K)} \Rightarrow R^{p+q}(N)^{\text{co}A(G)}.$$

We remark that Corollary 3.2 can be reformulated as follows: for a rational G -module N , we have a spectral sequence

$$H^p(K, H^q(L, N)) \Rightarrow H^{p+q}(G, N).$$

Here $H^p(G, -) = R^p(-)^G$ is the p -th right derived functor of the G -invariants functor $(-)^G$, from rational G -modules to vector spaces. Indeed, the category of rational G -modules is isomorphic to the category of $A(G)$ -comodules, and G -invariants of a rational G -module are precisely the $A(G)$ -coinvariants of an $A(G)$ -comodule. For more information on this spectral sequence, see [9, Theorem 2.9] and [4, Lemma 1.1].

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