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KATO MEETS BAKRY-ÉMERY

GILLES CARRON, ILARIA MONDELLO, AND DAVID TEWODROSE

Abstract. We prove that any complete Riemannian manifold with negative part of the Ricci curvature in a suitable Dynkin class is bi-Lipschitz equivalent to a finite-dimensional RCD space, by building upon the transformation rule of the Bakry-Émery condition under time change. We apply this result to show that our previous results on the limits of closed Riemannian manifolds satisfying a uniform Kato bound [CMT21, CMT22] carry over to limits of complete manifolds. We also obtain a weak version of the Bishop-Gromov monotonicity formula for manifolds satisfying a strong Kato bound.

1. Introduction

In a recent series of articles, we studied the structure of Gromov–Hausdorff limits of closed Riemannian manifolds with Ricci curvature satisfying some uniform Kato type condition [CMT21, CMT22]. The aim of this paper is to lift technical restrictions, like the closeness of the approximating manifolds, and to improve our previous results.

For a complete Riemannian manifold (M^n, g) of dimension $n \geq 2$, define

$$\mathrm{k}_t(M^n, g) := \sup_{x\in M} \int_0^t \int_M H(s, x, y)\mathrm{Ric}_-(y)\, \mathrm{d}\nu_g(y)\, \mathrm{d}s$$

for any $t > 0$, where H is the heat kernel of (M, g), ν_g is the Riemannian volume measure and $\mathrm{Ric}_- : M \to \mathbb{R}_+$ is the lowest non-negative function such that

$$\mathrm{Ric}_x \geq -\mathrm{Ric}_-(x)g_x$$

for any $x \in M$. From our previous work [CMT21, Corollary 2.5 and Theorem 4.10], a classical contradiction argument shows that for any $\varepsilon > 0$ there exists $\delta > 0$ depending on n and ε only such that if (M^n, g) is closed and satisfies

$$\mathrm{k}_T(M^n, g) \leq \delta$$

for some $T > 0$, then for any $p \in M$ there exists a pointed $\mathrm{RCD}(0, n)$ space (X, d, μ, x) such that

$$\mathsf{d}_{GH}\left(B^M_{\sqrt{T}}(p), B^X_{\sqrt{T}}(x)\right) \leq \varepsilon\sqrt{T}.$$

Here d_{GH} stands for the Gromov–Hausdorff distance. We briefly recall that for $K \in \mathbb{R}$ and $N \in [1, +\infty]$, an $\mathrm{RCD}(K, N)$ space is a metric measure space with a synthetic notion of Ricci curvature bounded below by K and dimension bounded above by N. The main result of this paper is a quantitative improvement of the previous fact:

Theorem A. *Let (M^n, g) be a complete Riemannian manifold. Assume that there exist $T > 0$ and $\gamma \in (0, 1/(n-2))$ such that*

$$\mathrm{k}_T(M^n, g) \leq \gamma. \tag{D}$$

Then there exist $K \leq 0$ and $N > n$, both depending on n and γ only, and $f \in \mathcal{C}^2(M)$ with $0 \leq f \leq C = C(n, \gamma)$, such that the weighted Riemannian manifold

$(M, e^{2f}g, e^{2f}\nu_g)$ *satisfies the* $\mathrm{RCD}(K/T, N)$ *condition. Moreover, if*

$$\mathrm{k}_T(M^n, g) \leq \frac{1}{3(n-2)} \tag{D'}$$

then we can choose $K = -4\mathrm{k}_T(M^n, g)$*,* $N = n + 4(n-2)^2\mathrm{k}_T(M^n, g)$ *and* $C = 4k_T(M^n, g)$*.*

As a consequence of Theorem A, we establish the existence of cut-off functions with controlled gradient and Laplacian, see Proposition 4.2. This implies that all the results proved in [CMT21, CMT22] on closed Riemannian manifolds extend to complete ones, see Section 4.

From the stability of the RCD condition under pointed measured Gromov–Hausdorff convergence, Theorem A also yields the following result for limit spaces.

Corollary B. *Let* (X, d, μ, o) *be the pointed measured Gromov–Hausdorff limit of a sequence of pointed complete weighted Riemannian manifolds* $\{(M^n_\ell, g_\ell, c_\ell \nu_{g_\ell}, o_\ell)\}$ *where* $\{c_\ell\} \subset (0, +\infty)$*. Assume that there exist* $T > 0$ *and* $\gamma \in (0, 1/(n-2))$ *such that*

$$\sup_\ell \mathrm{k}_T(M^n_\ell, g_\ell) \leq \gamma. \tag{UD}$$

Then there exist an $\mathrm{RCD}(K/T, N)$ *metric measure space* $(\overline{X}, \overline{\mathsf{d}}, \overline{\mu})$ *and a bi-Lipschitz map* $\Psi : X \to \overline{X}$ *such that* $\overline{\mu} \leq \Psi_\# \mu \leq \overline{\mu}$*. Here* K *and* N *are given by Theorem A.*

This corollary puts us in a position to apply well-known results of the RCD theory [DPMR17, KM18, MN19, BS20, GP21] to conclude that X is a rectifiable metric measure space with a constant essential dimension, see Propositions 5.4 and 5.7. This is a significant improvement over the rectifiability result that we proved in [CMT22, Theorem 4.4]. Corollary B also yields that the singularities of X are no more complicated than those of the boundary elements of the class of smooth $\mathrm{RCD}(K/T, N)$ spaces.

Another important consequence of Theorem A is an almost monotonicity formula for the volume ratio

$$\mathcal{V}(x, r) := \frac{\nu_g(B_r(x))}{\omega_n r^n}.$$

Here ω_n is the Lebesgue measure of the unit Euclidean ball in $\mathbb{R}^n$. To state that formula, let us consider a non-decreasing function $f : (0, T] \to \mathbb{R}_+$ such that

$$f(T) \leq \frac{1}{3(n-2)} \qquad \text{and} \qquad \int_0^T \frac{f(s)}{s}\, \mathrm{d}s < \infty. \tag{SK}$$

For any $\tau \in (0, \sqrt{T}]$ we set

$$\Phi(\tau) := \int_0^\tau \frac{f(s^2)}{s}\, \mathrm{d}s.$$

Theorem C. *Let* (M^n, g) *be a complete Riemannian manifold such that for any* $t \in (0, T]$*,*

$$\mathrm{k}_t(M^n, g) \leq f(t). \tag{K}$$

Then for any $x \in M$*,* $R \in (0, \sqrt{T}]$*,* $\eta \in (0, 1 - 1/\sqrt{2})$ *and* $r \leq (1-\eta)R$*,*

$$\mathcal{V}(x, R) \exp\left(-\frac{C(n)\Phi(R)}{\eta}\right) \leq \mathcal{V}(x, r) \exp\left(-\frac{C(n)\Phi(r)}{\eta}\right).$$

As a corollary, we obtain the following Hölder regularity result. We denote by $\mathcal{H}^n$ the n-dimensional Hausdorff measure of a metric space.

Corollary D. *Let (X, d, o) be the pointed Gromov–Hausdorff limit of a non-collapsed sequence of pointed complete Riemannian manifolds $\{(M_i^n, g_i, o_i)\}$ satisfying* (K). *Then the volume density*

$$\theta_X(x) \coloneqq \lim_{r\to 0+} \frac{\mathcal{H}^n(B_r(x))}{\omega_n r^n}$$

is well-defined at any $x \in X$, and for any $\alpha \in (0,1)$ there exists $\delta = \delta(n, \alpha, f) > 0$ such that the set $\{x \in X : \theta_X(x) \geq 1-\delta\}$ is contained in an open $\mathcal{C}^\alpha$ manifold.

Compare to [CMT22, Corollary 5.20] and note that the integral condition in (SK) is weaker than the strong Kato condition considered in that paper: see Remark 4.5 for the details.

To establish Theorem A, we adapt the proof of a classical result on Schrödinger operators to show that assumption (D) ensures the existence of a suitable gauge function $\varphi \in \mathcal{C}^2(M)$ that satisfies

$$\Delta_g \varphi - \lambda \mathrm{Ric}_- \varphi \geq -2\beta\varphi$$

for carefully chosen parameters $\lambda, \beta > 0$ depending on n and γ only. We then set

$$f \coloneqq \frac{1}{\lambda}\log \varphi.$$

With this choice of conformal factor, the transformation rule under time change ([Stu18, Stu20, HS21], see also Lemma 2.2) yields that $(M, e^{2f}g, e^{2f}\nu_g)$ satisfies a suitable version of the Bochner inequality, namely the Bakry-Émery condition $\mathrm{BE}(K/T, N)$, for K and N depending on n and γ only. The conclusion follows since $\mathrm{BE}(K/T, N)$ is equivalent to $\mathrm{RCD}(K/T, N)$ in the setting of weighted Riemannian manifolds, see e.g. [Stu06a, Theorem 4.9] or [LV09, Theorem 0.12]. The idea of using this transformation was inspired by [CMR22] where such a conformal change was made to prove a rigidity result for minimal hypersurfaces in $\mathbb{R}^4$ (see [CL21] for another proof of this result). This idea was implicity present already in [ENR07].

The paper is organised as follows. In Section 2, we recall the Bakry-Émery condition, time changes, and the aforementioned transformation rule. Section 3 is devoted to proving Theorem A. We give consequences of this theorem for complete Riemannian manifolds in Section 4 and for limit spaces in Section 5.

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2. The Bakry-Émery condition under time change

Let (M^n, g) be a complete Riemannian manifold of dimension $n \geq 2$. We write Δ_g for the non-negative Laplace–Beltrami operator of (M^n, g) defined by

$$\int_M \langle d\varphi, d\phi\rangle_g \, \mathrm{d}\nu_g = \int_M \varphi \Delta_g \phi \, \mathrm{d}\nu_g \tag{1}$$

for all $\varphi, \phi \in \mathcal{C}_c^\infty(M)$. We will also use Δ_g to denote the unique self-adjoint extension of the Laplace–Beltrami operator, which maps $\mathcal{C}_0^\infty(M)$ to $L^2(M, \nu_g)$. The heat kernel of (M^n, g) is the kernel of its heat semigroup $(e^{-t\Delta_g})_{t>0}$; in particular, for any $\phi \in \mathcal{C}_c^\infty(M)$ and $x \in M$,

$$\left(e^{-t\Delta_g}\phi\right)(x) = \int_M H(t,x,y)\phi(y)\,\mathrm{d}\nu_g(y).$$

We will say that $(M^n, g, \bar{\nu})$ is a weighted Riemannian manifold if $\bar{\nu}$ is a measure absolutely continuous with respect to ν_g with positive $\mathcal{C}^2$ Radon–Nikodym density. Such a space admits a weighted Laplacian L defined through the Green formula obtained upon replacing Δ_g by L and ν_g by $\bar{\nu}$ in (1).

For any Borel set $A \subset M$, we will write $\mathbf{1}_A$ for the characteristic function of A and $\mathbf{1}$ for the constant function equal to 1, that is, $\mathbf{1} = \mathbf{1}_M$. We will denote the spectrum by spec.

If $\mu, \bar{\mu}$ are two Borel measures on a metric space and $C > 0$, we shall write

$$\mu \leq C\bar{\mu}$$

to denote that μ is absolutely continuous with respect to $\bar{\mu}$ with Radon–Nikodym derivative lower than or equal to C $\bar{\mu}$-almost everywhere.

We will write $\mathbb{B}_r^n$ for the Euclidean ball of radius r centered at the origin of $\mathbb{R}^n$.

2.1. **The Bakry-Émery condition.** The Bochner formula for (M^n, g) states that for all $u \in \mathcal{C}^\infty(M)$,

$$\langle d\Delta_g u, du\rangle_g - \frac{1}{2}\Delta_g |du|_g^2 = |\nabla^g du|_g^2 + \mathrm{Ric}(du, du). \tag{2}$$

Introducing the function $\mathrm{Ric}_- : M \to \mathbb{R}_+$ defined by

$$\mathrm{Ric}_-(x) = \begin{cases} 0 & \text{if } \mathrm{Ric}_x \geq 0, \\ -\min \mathrm{spec}\,(\mathrm{Ric}_x) & \text{otherwise,} \end{cases} \tag{3}$$

this yields the so-called Bochner inequality:

$$\langle d\Delta_g u, du\rangle_g - \frac{1}{2}\Delta_g |du|_g^2 \geq \frac{(\Delta_g u)^2}{n} - \mathrm{Ric}_- |du|_g^2. \tag{4}$$

The Bakry-Émery condition is the analogue of (4) for weighted Riemannian manifolds.

Definition 2.1. For $K \in \mathbb{R}$ and $N \in [1, +\infty]$, we say that a weighted Riemannian manifold $(M^n, g, \bar{\nu})$ with associated weighted Laplacian L satisfies the Bakry-Émery condition $\mathrm{BE}(K, N)$ if for any $u \in \mathcal{C}^\infty(M)$,

$$\langle dLu, du\rangle_g - \frac{1}{2}L\left(|du|_g^2\right) \geq \frac{(Lu)^2}{N} + K\,|du|_g^2. \tag{5}$$

Introduced in the setting of Dirichlet forms in [BÉ85], this condition was the first milestone towards the definition and the study of metric measure spaces with a synthetic notion of Ricci curvature bounded from below by K and dimension bounded above by N.

2.2. **Time changes.** For any $f \in \mathcal{C}^2(M)$, define $\bar{g} := e^{2f}g$ and $\bar{\nu} := e^{2f}\nu_g$. The operator

$$L := e^{-2f}\Delta_g$$

is associated with the Dirichlet energy

$$\mathcal{E} : L^2(M, \bar{\nu}) \ni u \mapsto \int_M |du|_g^2 \,\mathrm{d}\nu_g \in [0, +\infty],$$

in the sense that, for any $u \in L^2(M, \bar{\nu})$,

$$\mathcal{E}(u) = \int_M (Lu)u \,\mathrm{d}\bar{\nu}.$$

The operator L is also the weighted Laplacian of the weighted Riemannian manifold $(M^n, \overline{g}, \bar{\nu})$. Indeed, for any $u \in \mathcal{C}^\infty(M)$,

$$|du|^2_{\overline{g}} = e^{-2f}|du|^2_g$$

thus

$$\int_M |du|^2_{\overline{g}} \ \mathrm{d}\bar{\nu} = \mathcal{E}(u).$$

The terminology "time change" comes from the fact that the Brownian motion on the weighted Riemannian manifold $(M^n, \overline{g}, \bar{\nu})$ is obtained from the Brownian motion of (M^n, g) only by a shift in time (see for instance [Stu18, Remark 8.3] and the references therein).

2.3. **Transformation rule.** The next lemma provides the transformation rule for the Bakry-Émery condition under time change. This rule is valid in a much more general setting, see for instance [Stu18, Stu20] and [HS21]. For completeness and because our notation is slightly different, we provide a detailed proof.

Lemma 2.2. Let (M^n, g) be a complete Riemannian manifold and $f \in \mathcal{C}^2(M)$. Set $\overline{g} := e^{2f}g$, $\bar{\nu} := e^{2f}\nu_g$ and $L := e^{-2f}\Delta_g$. Then for any $q \in (0, +\infty]$ and $u \in \mathcal{C}^\infty(M)$,

$$\langle dLu, du\rangle_{\overline{g}} - \frac{1}{2}L|du|^2_{\overline{g}} \geq \frac{(Lu)^2}{n+q} + \left(-\mathrm{Ric}_- + \Delta_g f - c(n,q)|df|^2_g\right) e^{-2f}|du|^2_{\overline{g}}$$

where $c(n,q) = \frac{(n-2)(n+q-2)}{q}$.

Proof of Lemma 2.2. Recall the calculus rules:

$$\Delta_g(\varphi\phi) = \varphi\Delta_g\phi + \phi\Delta_g\varphi - 2\langle d\varphi, d\phi\rangle, \tag{6}$$

$$\Delta_g(\chi \circ \varphi) = (\chi' \circ \varphi)\Delta_g\varphi - (\chi'' \circ \varphi)|d\varphi|^2. \tag{7}$$

Using them, we easily compute that

$$\langle dLu, du\rangle_{\overline{g}} = e^{-4f}\langle d\Delta_g u, du\rangle_g - 2e^{-4f}\langle df, du\rangle_g \, \Delta_g u\,,$$

$$L|du|^2_{\overline{g}} = e^{-4f}\Delta_g|du|^2_g - 2e^{-4f}\Delta_g f - 4e^{-4f}|df|^2_g + 8e^{-4f}\nabla^g du(du, df)\,,$$

so that

$$e^{4f}\left(\langle dLu, du\rangle_{\overline{g}} - \frac{1}{2}L|du|^2_{\overline{g}}\right) = \langle d\Delta_g u, du\rangle_g - \frac{1}{2}\Delta_g|du|^2_g + \Delta_g f\,|du|^2_g + 2|df|^2_g|du|^2_g - 4\nabla^g du(du, df) - 2\langle df, du\rangle_g\Delta_g u.$$

Let

$$A := \nabla^g du + \frac{\Delta_g u}{n} g$$

be the traceless part of $\nabla^g du$. We introduce the tensor

$$B := (df \otimes du + du \otimes df)$$

whose traceless part is

$$\mathring{B} = B - \frac{\mathrm{tr}_g B}{n} g = B - 2\frac{\langle du, df\rangle_g}{n} g.$$

Using the Bochner formula (2), we get that

$$\begin{aligned} e^{4f}\left(\langle dLu, du\rangle_{\overline{g}} - \frac{1}{2}L|du|^2_{\overline{g}}\right) &= |A|^2 - 2\langle A, \mathring{B}\rangle \\ &\quad + \mathrm{Ric}(du, du) + \Delta_g f|du|^2_g \\ &\quad + 2|df|^2_g|du|^2_g + \frac{(\Delta_g u)^2}{n} \\ &\quad + \frac{4}{n}\langle du, df\rangle_g\Delta_g u - 2\langle df, du\rangle_g\Delta_g u. \end{aligned}$$

Then using that

$$|A|^2 - 2\langle A, \mathring{B}\rangle \geq -|\mathring{B}|^2 = -2\left(|df|_g^2|du|_g^2 + \langle du, df\rangle_g^2\right) + \frac{4}{n}\langle du, df\rangle_g^2,$$

we eventually obtain

$$\begin{aligned} e^{4f}\left(\langle dLu, du\rangle_{\overline{g}} - \frac{1}{2}L|du|_{\overline{g}}^2\right) \geq \frac{(\Delta_g u)^2}{n} \\ + \left(-\mathrm{Ric}_- + \Delta_g f\right)|du|_g^2 \\ - \left(2 - \frac{4}{n}\right)\langle du, df\rangle_g^2 \\ - 2\,\frac{n-2}{n}\langle du, df\rangle_g \Delta_g u. \end{aligned}$$

By the Young inequality, we have

$$-2\,\frac{n-2}{n}\langle du, df\rangle_g\Delta_g u \geq -\left(\frac{1}{n} - \frac{1}{n+q}\right)(\Delta_g u)^2 - \frac{(n-2)^2 n(n+q)}{n^2 q}\langle du, df\rangle_g^2,$$

hence the Cauchy-Schwarz inequality and some simple computations yield the desired inequality. □

Corollary 2.3. Let (M^n, g) be a Riemannian manifold and $\varphi \in \mathcal{C}^2(M)$ such that $\varphi \geq 1$ and

$$\Delta\varphi - \lambda \mathrm{Ric}_-\varphi \geq -\kappa\varphi$$

for some $\lambda > n-2$ and $\kappa \geq 0$. If we set $f = \frac{1}{\lambda}\log\varphi$, then the weighted Riemannian manifold $(M^n, e^{2f}g, e^{2f}\nu_g)$ satisfies the $\mathrm{BE}(\kappa/\lambda, n+q)$ condition, where $q = \dfrac{(n-2)^2}{\lambda - (n-2)}$.

Proof. This is a direct consequence of Lemma 2.2 and the chain rule (7):

$$\Delta_g f = \frac{\Delta_g\varphi}{\lambda\varphi} + \frac{1}{\lambda}\frac{|d\varphi|_g^2}{\varphi^2} = \frac{\Delta_g\varphi}{\lambda\varphi} + \lambda|df|_g^2 = \frac{\Delta_g\varphi}{\lambda\varphi} + \frac{(n-2)(n+q-2)}{q}|df|_g^2.$$ □

3. Proof of Theorem A

3.1. **Kato condition and the bottom of the spectrum.** In this subsection, we recall a useful fact about Schrödinger operators whose potential satisfies a so-called Dynkin condition. Let (M^n, g) be a complete Riemannian manifold and $V \geq 0$ a locally integrable function on M. For any $t > 0$, we define

$$\mathrm{k}_t(V) := \sup_{x\in M}\iint_{[0,t]\times M} H(s,x,y)V(y)\,\mathrm{d}\nu_g(y)\,\mathrm{d}s.$$

It is classical (see e.g. [Gün17]) that if V satisfies the Dynkin condition

$$\mathrm{k}_T(V) < 1,$$

then the quadratic form

$$\mathcal{C}_c^\infty(M) \ni u \mapsto \int_M \left(|du|_g^2 - Vu^2\right)\mathrm{d}\nu_g$$

is bounded from below on $L^2(M, \nu_g)$, hence it generates a self-adjoint operator $H_V = \Delta_g - V$ whose heat semi-group $\left\{e^{-tH_V}\right\}_{t>0}$ acts boundedly on each $L^p(M, \nu_g)$. More precisely, for any $p \in [1, +\infty]$ there exist $C > 0$ and $\omega \geq 0$ such that for any $t \geq 0$,

$$\left\|e^{-tH_V}\right\|_{L^p\to L^p} \leq Ce^{\omega t}.$$

The proof of this classical result yields more precise information:

Proposition 3.1. Let (M,g) be a complete Riemannian manifold. Let $V \geq 0$ be a locally integrable function on M such that for some $T, \beta > 0$,

$$\mathrm{k}_T(V) \leq 1 - e^{-\beta T}.$$

Then:

i) for any $\phi \in \mathcal{C}_c^\infty(M)$,

$$\int_M \left[|d\phi|^2 - V\phi^2 \right] \mathrm{d}\nu_g \geq -\beta \int_M \phi^2 \, \mathrm{d}\nu_g;$$

ii) $\operatorname{spec} H_V \subset [-\beta, +\infty)$;

iii) for any $p \in [1, +\infty]$ and $t \geq 0$:

$$\left\| e^{-tH_V} \right\|_{L^p \to L^p} \leq e^{\beta(t+T)}.$$

Proof. Note that ii) follows from i) and the min-max characterization of the elements of $\operatorname{spec} H_V$. Moreover, i) is a consequence of the case $p = 2$ in iii). Therefore, we need only to prove iii).

For any $\ell \in \mathbb{N}$, set $V_\ell := \min(V, \ell)\, \mathbf{1}_{B_\ell(o)}$ and note that $\mathrm{k}_T(V_\ell) \leq \mathrm{k}_T(V)$. If $p \geq 0$ is such that

$$\left\| e^{-tH_{V_\ell}} \right\|_{L^p \to L^p} \leq e^{\beta(t+T)}$$

for any ℓ and $t > 0$, then the monotone convergence theorem ensures that e^{-tH_V} is well-defined as the pointwise limit of $\{e^{-tH_{V_\ell}}\}$, and $\left\| e^{-tH_V} \right\|_{L^p \to L^p} \leq e^{\beta(t+T)}$. Therefore, from now on we assume that V is bounded with bounded support.

Using selfadjointness and the Schur test, we need only proving that for any $t \geq 0$,

$$M_V(t) := \left\| e^{-tH_V} \mathbf{1} \right\|_{L^\infty} \leq e^{\beta(t+T)}.$$

We already have that

$$e^{-tH_V} \mathbf{1} \geq \mathbf{1},$$

hence by the semi-group law $t \mapsto M_V(t)$ is non-decreasing. The Duhamel formula implies that

$$e^{-tH_V} \mathbf{1} = e^{-t\Delta_g} \mathbf{1} + \int_0^t e^{-(t-s)\Delta_g} V e^{-sH_V} \mathbf{1} \, \mathrm{d}s.$$

But $e^{-t\Delta_g} \mathbf{1} \leq \mathbf{1}$ hence

$$M_V(t) \leq 1 + \left(\sup_{x \in M} \iint_{[0,t] \times M} H(t-s, x, y) V(y) \, \mathrm{d}\nu_g(y) \, \mathrm{d}s \right) \sup_{s \in [0,t]} M_V(s).$$

Consequently, when $t \in [0, T]$, we get that

$$M_V(t) \leq 1 + \left(1 - e^{-\beta T}\right) M_V(t),$$

which leads to

$$M_V(t) \leq M_V(T) \leq e^{\beta T}.$$

Let $t \in [kT, (k+1)T]$. Using the semi-group law, one gets that

$$M_V(t) \leq M_V\left((k+1)T\right) \leq \left(M_V(T)\right)^{k+1} \leq e^{\beta(k+1)T} \leq e^{\beta(t+T)}. \qquad \square$$

The previous proposition easily yields that φ in the next corollary is well-defined and satisfies the desired properties.

Corollary 3.2. Let (M,g) be a complete Riemannian manifold. Let $V \geq 0$ be a locally integrable function on M such that for some $T, \beta > 0$,

$$\mathrm{k}_T(V) \leq 1 - e^{-\beta T}.$$

Then the function

$$\varphi := 2\beta \int_0^{+\infty} e^{-2\beta t} e^{-tH_V} \mathbf{1} \, \mathrm{d}t$$

satisfies $1 \leq \varphi \leq 2e^{\beta T}$ and

$$H_V \varphi + 2\beta\varphi = 2\beta.$$

Remark 3.3. In the previous corollary, elliptic regularity implies that if V is $\mathcal{C}^{k,\alpha}$ for some $\alpha \in (0,1)$, then φ is $\mathcal{C}^{k+2,\alpha}$.

3.2. **Proof of Theorem A.** We are now in a position to prove Theorem A.

Proof. Let (M^n, g) be a complete Riemannian manifold satisfying (D). Set

$$\lambda \coloneqq \frac{1}{2}\left(n - 2 + \gamma^{-1}\right)$$

and consider $\beta > 0$ such that

$$e^{-\beta T} = \frac{1}{2}\left(1 - (n-2)\gamma\right).$$

Since $\lambda > n-2$ and Ric_- is a continuous function, Corollary 3.2 and Remark 3.3 ensure that there exists $\varphi \in \mathcal{C}^2(M)$ such that $1 \leq \varphi \leq 2e^{\beta T}$ and

$$\Delta_g \varphi - \lambda \mathrm{Ric}_- \varphi \geq -2\beta\varphi.$$

Define

$$f \coloneqq \frac{1}{\lambda}\log\varphi.$$

Then Corollary 2.3 implies that the weighted Riemanniann manifold $(M^n, e^{2f}g, e^{2f}\nu_g)$ satisfies the $\mathrm{BE}(-2\beta/\lambda, n+q)$ condition with $q = \dfrac{2(n-2)^2\gamma}{1-(n-2)\gamma}\cdot$ Setting

$$K = K(n,\gamma) \coloneqq -\frac{4\ln(\frac{1}{2}(1-(n+2)\gamma))}{(n-2+\gamma^{-1})} \quad \text{and} \quad N = N(n,\gamma) \coloneqq n+q,$$

we get that $(M^n, e^{2f}g, e^{2f}\nu_g)$ satisfies the $\mathrm{BE}(-K/T, N)$ condition.

Assume now that (D') holds. We make a different choice for the parameters β and λ, namely

$$\beta \coloneqq 1/T \qquad \text{and} \qquad \lambda \coloneqq \frac{1-e^{-1}}{\mathrm{k}_T(M^n,g)},$$

so that $q > 0$ is given by

$$q = \frac{(n-2)^2 \mathrm{k}_T(M^n,g)}{1 - e^{-1} - (n-2)\mathrm{k}_T(M^n,g)}\cdot$$

Since $1 - e^{-1} \geq \frac{1}{2}$ and $1 - e^{-1} - \frac{1}{3} \geq \frac{1}{4}$ we get that

$$\frac{1}{\lambda} \leq 2\mathrm{k}_T(M^n,g) \quad \text{and} \quad q \leq 4(n-2)^2\mathrm{k}_T(M^n,g),$$

and then

$$0 \leq f \leq \frac{1}{\lambda}\ln(2e) \leq \frac{2}{\lambda} \leq 4\mathrm{k}_T(M^n,g). \qquad \square$$

4. Consequences on complete manifolds

4.1. **Almost mononicity of the volume ratio.** Theorem C is a direct consequence of the following proposition.

Proposition 4.1. Let (M^n, g) be a complete Riemannian manifold satisfying (D') for some $T > 0$. Then for any $x \in M$, $\eta \in (0, 1 - 1/\sqrt{2}]$, $R \in (0, \sqrt{T}]$ and $r \in (0, (1-\eta)R]$,

$$\frac{\nu_g\left(B_R(x)\right)}{\nu_g\left(B_r(x)\right)} \leq \left(\frac{R}{r}\right)^n \exp\left(\frac{C(n)}{\log(1/(1-\eta))}\int_r^R \frac{\mathrm{k}_{s^2}(M^n,g)}{s}\,\mathrm{d}s\right)\cdot \tag{8}$$

In order to prove this result, we must recall some well-known facts. Consider $\kappa \geq 0$ and $N \in [1, +\infty)$. As shown in [Qia97, Lot03, BQ05, WW09], the Bishop–Gromov comparison theorem holds on any complete weighted Riemannian manifold $(M, g, \bar{\nu})$ satisfying the $\mathrm{BE}(-\kappa, N)$ condition: for any $x \in M$ and $0 < r < R$,

$$\frac{\bar{\nu}\left(B_R(x)\right)}{\bar{\nu}\left(B_r(x)\right)} \leq \frac{\mathbb{V}_{\kappa,N}(R)}{\mathbb{V}_{\kappa,N}(r)}$$

where

$$\mathbb{V}_{\kappa,N}(\rho) \coloneqq \int_0^\rho \sinh^{N-1}(\kappa s)\ \mathrm{d}s$$

for any $\rho > 0$. Using the inequality

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{1}{\sinh(\sigma)}\right) = \frac{\cosh(\sigma)}{\sinh(\sigma)} \leq \frac{1}{\sigma} + \frac{\sigma}{2}$$

which holds for any $\sigma > 0$, we can classically bound the previous right-hand side from above and get the following estimate

$$\frac{\bar{\nu}\left(B_R(x)\right)}{\bar{\nu}\left(B_r(x)\right)} \leq e^{\bar{C}(N)\kappa^2 R^2}\left(\frac{R}{r}\right)^N, \tag{9}$$

with $\bar{C}(N) \coloneqq (N-1)/4$.

Proof of Proposition 4.1. Consider $x \in M$ and $\eta \in (0, 1 - 1/\sqrt{2})$. Define $\lambda(\tau) \coloneqq \mathrm{k}_{\tau^2}(M^n, g)$ for any $\tau > 0$.

Let us first show that for any $0 < r \leq \sqrt{T}$ and $\rho > r$,

$$\frac{\nu_g\left(B_{e^{-4\lambda(r)}\rho}(x)\right)}{\nu_g\left(B_r(x)\right)} \leq \left(\frac{\rho}{r}\right)^n \exp\left(C(n)\lambda(r)\left(\frac{\rho^2}{r^2} + \log\left(\frac{\rho}{r}\right) + 1\right)\right). \tag{10}$$

Since λ is non-decreasing, the assumption (D') implies that for any $r \in (0, \sqrt{T})$,

$$\lambda(r) \leq \frac{1}{3(n-2)}.$$

According to Theorem A, there exists $f \in \mathcal{C}^2(M)$ with

$$0 \leq f \leq 4\lambda(r) \tag{11}$$

such that the weighted Riemannian manifold $(M^n, \overline{g} \coloneqq e^{2f}g, \bar{\nu} \coloneqq e^{2f}\nu_g)$ satisfies the $\mathrm{BE}(-4\lambda(r)/r^2, n + 4(n-2)^2\lambda(r))$ condition. Using an overline to denote the geodesic balls of the metric $\overline{g}$, by (9) we get that for any $\rho > r$,

$$\frac{\bar{\nu}\left(\overline{B}_\rho(x)\right)}{\bar{\nu}\left(\overline{B}_r(x)\right)} \leq \left(\frac{\rho}{r}\right)^n \exp\left(C(n)\lambda(r)\left(\frac{\rho^2}{r^2} + \log\left(\frac{\rho}{r}\right)\right)\right). \tag{12}$$

From (11) we deduce that

$$\overline{B}_r(x) \subset B_r(x), \quad B_{e^{-4\lambda(r)}\rho}(x) \subset \overline{B}_\rho(x) \quad \text{and} \quad \nu_g \leq \bar{\nu} \leq e^{8\lambda(r)}\nu_g,$$

which easily lead to (10) from (12).

We are now in a position to prove (8) for $R \in (0, \sqrt{T}]$ and $r \in [R/2, (1-\eta)R]$. Apply (10) with $\rho = Re^{4\lambda(r)}$:

$$\frac{\nu_g\left(B_R(x)\right)}{\nu_g\left(B_r(x)\right)} \leq \left(\frac{R}{r}\right)^n \exp\left(C(n)\lambda(r)\left(\frac{R^2}{r^2} + \log\left(\frac{R}{r}\right) + 1\right)\right).$$

Since $r \geq R/2$, we deduce that[1]

$$\frac{\nu_g\left(B_R(x)\right)}{\nu_g\left(B_r(x)\right)} \leq \left(\frac{R}{r}\right)^n \exp\left(C(n)\lambda(r)\right).$$

[1] changing $C(n)(4 + \log(2) + 1)$ into $C(n)$

Using that $r \leq (1-\eta)R$ and λ is non-decreasing, we get

$$\lambda(r) \leq \frac{\lambda(r)}{\log(1/(1-\eta))}\int_r^R \frac{\mathrm{d}s}{s} \leq \frac{1}{\log(1/(1-\eta))}\int_r^R \lambda(s)\frac{\mathrm{d}s}{s}$$

so that (8) is proved.

To conclude, it remains to consider the case $r \in (0, R/2)$. Set $r_k := (1-\eta)^{-k}r$ for any $k \in \mathbb{N}$. Let ℓ be the integer such that

$$(1-\eta)^2 R < r_l \leq (1-\eta)R < \sqrt{T}.$$

Note that $r_l \in [R/2, (1-\eta)R]$ because $R/2 < (1-\eta)^2 R$. Moreover, since

$$(1-\eta)r_k = r_{k-1} \geq (1-\eta)^2 r_k \geq r_k/2,$$

we have $r_{k-1} \in [r_k/2, (1-\eta)r_k]$ for any $k \in \{1,\dots,\ell\}$. Therefore, the previous argument yields that

$$\frac{\nu_g\left(B_R(x)\right)}{\nu_g\left(B_{r_\ell}(x)\right)} \leq \left(\frac{R}{r_\ell}\right)^n \exp\left(\frac{C(n)}{\log(1/(1-\eta))}\int_{r_\ell}^R \lambda(s)\frac{\mathrm{d}s}{s}\right)$$

and

$$\frac{\nu_g\left(B_{r_k}(x)\right)}{\nu_g\left(B_{r_{k-1}}(x)\right)} \leq \left(\frac{r_k}{r_{k-1}}\right)^n \exp\left(\frac{C(n)}{\log(1/(1-\eta))}\int_{r_{k-1}}^{r_k} \lambda(s)\frac{\mathrm{d}s}{s}\right)$$

for any $k \in \{1,\dots,\ell\}$, and (8) follows by taking the product of all these inequalities. □

4.2. **Existence of good cut-off functions.**

Proposition 4.2. Let (M^n, g) be a complete Riemannian manifold satisfying (D') for some $T > 0$. Then for any $x \in M$ and $r \in (0, \sqrt{T})$, there exists $\chi_{x,r} \in \mathcal{C}^4(M)$ such that:

i) $\chi_{x,r} = 1$ on $B_{r/2}(x)$ and $\chi_{x,r} = 0$ outside $B_r(x)$,

ii) $|d\chi_{x,r}|_g^2 + |\Delta_g \chi_{x,r}| \leq \dfrac{C(n)}{r^2}\cdot$

Proof. By Theorem A, there exists $f \in \mathcal{C}^2(M)$ with

$$0 \leq f \leq 4\mathrm{k}_T(M^n, g) \leq 4/3 \tag{13}$$

such that $\left(M, \overline{g} := e^{2f}g, \overline{\nu} := e^{2f}\nu_g\right)$ is an $\mathrm{RCD}(-1/T, 3n)$ space. Let $\overline{\mathsf{d}}$ be the Riemannian distance associated with $\overline{g}$ and $\overline{B}_r(x)$ the $\overline{g}-$geodesic ball centered at $x \in M$ with radius $r > 0$. By (13), we have

$$\mathsf{d}_g \leq \overline{\mathsf{d}} \leq e^{\frac{4}{3}}\mathsf{d}_g.$$

Since $e^{\frac{4}{3}} \leq 4$ we get the inclusions

$$B_{r/4}(x) \subset \overline{B}_r(x) \subset B_r(x) \tag{14}$$

for any $x \in M$ and $r > 0$. According to [MN19, Lemma 3.1], for any $z \in M$ and $r \in (0, \sqrt{T})$ there exists $\phi_{z,r} \in \mathcal{C}^4(M)$ such that:

- $\phi_{z,r} = 1$ on $\overline{B}_{r/2}(z)$ and $\phi_{z,r} = 0$ outside $\overline{B}_r(z)$,
- $|d\phi_{z,r}|_{\overline{g}}^2 + |L\phi_{z,r}| \leq \dfrac{C(n)}{r^2}\cdot$

Now (13) yields

$$|d\phi_{z,r}|_g^2 + |\Delta_g \phi_{z,r}| \leq \frac{C(n)}{r^2}$$

for some new constant $C(n)$, and (14) implies that $\phi_{z,r} = 1$ on $B_{r/8}(z)$ and $\phi_{z,r} = 0$ outside of $B_r(z)$.

Now, consider $x \in M$ and $r \in (0, \sqrt{T})$. Let $\{z_i\}_{i\in I} \subset B_{r/2}(x)$ be such that the balls $\overline{B}_{r/(32)}(z_i)$ are disjoint one to another and $B_{r/2}(x) \subset \cup_i \overline{B}_{r/(16)}(z_i)$. The Bishop–Gromov comparison theorem on $(M, \overline{g}, \overline{\nu})$ classically implies that there is an integer $N(n)$ such that $\#I \leq N(n)$. Set

$$\xi_{x,r} \coloneqq \sum_i \phi_{z_i, r/8}.$$

Then by construction $\xi_{x,r} \geq 1$ on $B_{r/2}(x)$. Moreover $\xi_{x,r}$ is zero outside

$$\cup_i \overline{B}_{r/8}(z_i) \subset \cup_i B_{r/2}(z_i) \subset B_r(x).$$

We easily get the estimate

$$|d\xi_{x,r}|_g^2 + |\Delta_g \xi_{x,r}| \leq \frac{C(n)N^2(n)}{r^2} \cdot$$

Eventually, we set

$$\chi_{x,r} \coloneqq u \circ \xi_{x,r}$$

where $u \in \mathcal{C}^\infty(\mathbb{R})$ is some fixed function such that $u = 1$ on $[1, +\infty)$ and $u = 0$ on $(-\infty, 0]$. □

Remark 4.3. The same proof also shows that if

$$\mathrm{k}_T(M^n, g) \leq \gamma < \frac{1}{n-2}$$

then for any $x \in M$ and $r \in (0, \sqrt{T})$, there exists $\chi_{x,r} \in \mathcal{C}^4(M)$ such that

i) $\chi_{x,r} = 1$ on $B_{r/2}(x)$ and $\chi_{x,r} = 0$ outside $B_r(x)$,

ii) $|d\chi_{x,r}|_g^2 + |\Delta_g \chi_{x,r}| \leq \dfrac{C(n,\gamma)}{r^2} \cdot$

As mentioned in the introduction and in [CMT21, Remark 3.4], the existence of cut-off functions like in the previous proposition implies that all the results of [CMT21, CMT22] extend to complete Riemannian manifolds. Indeed, our previous work relies, among others, on a Li–Yau type inequality: the restriction to the case of closed Riemannian manifolds was then due to the fact that this inequality [Car20, Proposition 3.3] was proved only for closed manifolds. Moreover, it is known that a complete Riemannian manifold (M^n, g) with cut-off functions as above and such that $\mathrm{k}_T(M^n, g) \leq 1/(16n)$ satisfies the same Li–Yau type inequality (see [Car20, Proposition 3.16]), this allowing to apply our results in the complete setting. We will not state all the results of [CMT21, CMT22] that now hold true on complete Riemannian manifolds but we will focus on some key results.

4.2.1. *Monotonicity of heat ratios.* Let (M^n, g) be a complete Riemannian manifold. For any $t > 0$ and $x, y \in M$, set

$$U(t,x,y) \coloneqq -4t \log\left((4\pi t)^{\frac{n}{2}} H(t,x,y)\right).$$

Theorem 4.4. Let (M^n, g) be a complete Riemannian manifold such that for some $T > 0$,

$$\mathrm{k}_T(M^n, g) \leq \frac{1}{16n} \quad \text{and} \quad \int_0^T \frac{\mathrm{k}_t(M^n, g)}{t}\, \mathrm{d}t < \infty.$$

For any $t \in (0, T)$, set

$$\Phi(t) \coloneqq \int_0^t \frac{\mathrm{k}_\tau(M^n, g)}{\tau}\, \mathrm{d}\tau.$$

Then for any $t \in (0,T)$ and $s > 0$ there exists $\overline{\lambda} = \overline{\lambda}(n, \Phi(T), s/t) > 0$ such that $\lim_{\sigma \to 0+} \overline{\lambda}(n, \Phi(T), \sigma) = 0$ and the function

$$\lambda \in (0, \overline{\lambda}] \mapsto e^{c_n \Phi(\lambda t)\left(\frac{t}{s} - \frac{s}{t}\right)} \int_M \frac{e^{-\frac{U(\lambda t, x, y)}{4\lambda s}}}{(4\pi \lambda s)^{\frac{n}{2}}} \, \mathrm{d}\nu_g(y)$$

is monotone. It is non-increasing if $s \geq t$ and non-decreasing if $s \leq t$.

Remark 4.5. In [CMT21, Corollary 5.10] we used the Li–Yau type inequality to prove the previous monotonicity under the assumptions

$$\mathrm{k}_T(M^n, g) \leq \frac{1}{16n} \quad \text{and} \quad \int_0^T \frac{\sqrt{\mathrm{k}_t(M^n, g)}}{t} \, \mathrm{d}t < \infty.$$

A close look at the proof of [Car20, Proposition 3.3] shows that a different choice of the parameters δ, α, namely

$$\delta \simeq (\mathrm{k}_T(M^n, g))^2 \text{ and } \alpha = 1 - \sqrt{\frac{n\delta}{2 - \delta}},$$

leads to a Li–Yau inequality where the term $\sqrt{\mathrm{k}_T(M^n, g)}$ is replaced by a multiple of $\mathrm{k}_T(M^n, g)$. Theorem 4.4 is then obtained by using this latter version of the Li–Yau inequality in the proof of [CMT21, Corollary 5.10].

4.2.2. *Local doubling and Poincaré.* As [Car20] shows, the validity of the Li–Yau inequality on a complete Riemannian manifold satisfying

$$\mathrm{k}_T(M^n, g) \leq \frac{1}{16n}$$

implies that (M^n, g) is locally doubling and satisfies the local Poincaré inequality. Below, we prove these properties differently, using the fact that they are preserved under a bi-Lipschitz change of the metric and the measure.

Proposition 4.6. Let (M^n, g) be a complete Riemannian manifold satisfying (D). Then there exist C, N, λ depending only on n and γ such that for any $x \in M$ and $0 < r \leq \sqrt{T}$,

(1) for any $s \in (0, r)$,

$$\nu_g(B_r(x)) \leq C \left(\frac{r}{s}\right)^N \nu_g(B_s(x));$$

(2) for any $\varphi \in \mathcal{C}^1(B_r(x))$ with $\int_{\overline{B}_r(x)} \varphi \, \mathrm{d}\bar{\nu} = 0$,

$$(\star) \quad \|\varphi\|_{L^1(B_r(x))} \leq \lambda r \, \|d\varphi\|_{L^1(B_r(x))}.$$

Proof. According to Theorem A there exist C, K, N, depending only on n and γ, and $f \in \mathcal{C}^2(M)$ with

$$1 \leq e^f \leq C,$$

such that the weighted Riemannian manifold $\left(M^n, \overline{g} := e^{2f} g, \bar{\nu} := e^{2f} \nu_g\right)$ satisfies the $\mathrm{BE}\left(\frac{K}{T}, N\right)$ condition. Hence the Bishop-Gromov inequality (9) implies that the $\bar{\nu}$-measure of the $\overline{g}$-geodesic ball satisfies :

$$\forall x \in M, 0 < r < R \colon \bar{\nu}\left(\overline{B}_R(x)\right) \leq e^{C(n,\gamma)\frac{R^2}{T}} \left(\frac{R}{r}\right)^N \bar{\nu}\left(\overline{B}_r(x)\right);$$

We then have for $0 < s \leq r \leq \sqrt{T}$ and $x \in M$:

$$\begin{aligned}\nu_g\left(B_r(x)\right) &\leq \bar{\nu}\left(B_r(x)\right)\\ &\leq \bar{\nu}\left(\overline{B}_{Cr}(x)\right)\\ &\leq C^N e^{C(n,\gamma)\frac{r^2}{T}}\left(\frac{r}{s}\right)^N \bar{\nu}\left(\overline{B}_s(x)\right)\\ &\leq C^{N+2} e^{C(n,\gamma)}\left(\frac{r}{s}\right)^N \nu\left(\overline{B}_s(x)\right)\\ &\leq C^{N+2} e^{C(n,\gamma)}\left(\frac{r}{s}\right)^N \nu\left(B_s(x)\right).\end{aligned}$$

According to [Stu06b, LV07, vR08], see also [Vil09, Corollary 19.13], we also have the L^1-Poincaré inequality : if $x \in M$ and $r > 0$ then for any $\varphi \in \mathcal{C}^1\left(\overline{B}_r(x)\right)$ with $\int_{\overline{B}_r(x)} \varphi \, \mathrm{d}\bar{\nu} = 0$,

$$\int_{\overline{B}_r(x)} |\varphi| \, \mathrm{d}\bar{\nu} \leq C(n,\gamma) e^{C(n,\gamma)\frac{r^2}{T}} \, r \int_{\overline{B}_r(x)} |d\varphi|_{\overline{g}} \, \mathrm{d}\bar{\nu}.$$

Then if $\varphi \in \mathcal{C}^1\left(B_{Cr}(x)\right)$ with $c = \fint_{\overline{B}_r(x)} \varphi \, \mathrm{d}\bar{\nu}$ one gets

$$\begin{aligned}\|\varphi - c\|_{L^1(B_r(x),\nu)} &\leq \int_{\overline{B}_{Cr}(x)} |\varphi - c| \, \mathrm{d}\bar{\nu}\\ &\leq C(n,\gamma) e^{C(n,\gamma)\frac{r^2}{T}} \, r \int_{\overline{B}_{Cr}(x)} |d\varphi|_{\overline{g}} \, \mathrm{d}\bar{\nu}\\ &\leq C(n,\gamma)^2 e^{C(n,\gamma)\frac{r^2}{T}} \, r \int_{B_{Cr}(x)} |d\varphi|_g \, \mathrm{d}\nu.\end{aligned}$$

Moreover we always have

$$\left\|\varphi - \varphi_{B_r(x)}\right\|_{L^1(B_r(x))} \leq 2 \, \|\varphi - c\|_{L^1(B_r(x))}.$$

In order to conclude that we get the stronger Poincaré inequality ($\star$), we refer to the work of Jerison and of Maheux, Saloff-Coste [Jer86, MSC95]. □

5. Consequences for limit spaces

In this section, we explain how the previous results broaden the study carried out in [CMT21, CMT22] on limits of Riemannian manifolds with suitable uniform bounds on the Ricci curvature. We begin with a convenient definition.

Definition 5.1. We say that a pointed metric measure space (X, d, μ, o) is a renormalized limit space if it is the pointed measured Gromov–Hausdorff limit of a sequence of pointed complete weighted Riemannian manifolds $\{(M_\ell, g_\ell, \mu_\ell := c_\ell \nu_{g_\ell}, o_\ell)\}$ of same dimension such that there exists $\kappa > 0$ satisfying that, for any ℓ,

$$\kappa^{-1} c_\ell \leq \nu_{g_\ell}\left(B_{\sqrt{T}}(o_\ell)\right) \leq \kappa c_\ell. \tag{15}$$

Remarks 5.2.

1. We may denote a renormalized limit space by $(X, \mathsf{d}, \mu, o) \leftarrow (M^n_\ell, g_\ell, \mu_\ell, o_\ell)$ if needed.
2. A common renormalization is $c_\ell = \nu_{g_\ell}\left(B_{\sqrt{T}}(o_\ell)\right)^{-1}$ for all ℓ.

5.1. Dynkin limit spaces.

Consider a sequence $\{(M_\ell, g_\ell, \mu_\ell, o_\ell)\}$ as in Definition 5.1 and assume that it additionnally satisfies (UD) for some uniform $T > 0$ and $\gamma \in (0, 1/(n-2))$. Then the doubling condition from Proposition 4.6 and Gromov's compactness theorem imply that $\{(M_\ell, g_\ell, \mu_\ell, o_\ell)\}$ admits limit points in the pointed measured Gromov–Hausdorff topology. This leads to the following natural definition.

Definition 5.3. A renormalized limit space $(X, \mathsf{d}, \mu, o) \leftarrow (M^n_\ell, g_\ell, \mu_\ell, o_\ell)$ is called a Dynkin limit space if there exist $T > 0$ and $\gamma \in (0, 1/(n-2))$ such that (UD) holds.

We prove the following result for Dynkin limit spaces.

Proposition 5.4. Any Dynkin limit space (X, d, μ, o) is k-rectifiable for some integer k, in the sense that there exists a countable collection $\{(V_i, \phi_i)\}_i$ such that $\{V_i\}$ are Borel subsets covering X up to a μ-negligible set and $\phi_i : V_i \to \mathbb{R}^k$ is a bi-Lipschitz map satisfying $(\phi_i)_\#(\mu \llcorner V_i) \ll \mathcal{H}^k$ for any i.

Proof. Let $(X, \mathsf{d}, \mu, o) \leftarrow (M^n_\ell, g_\ell, \mu_\ell = c_\ell \nu_{g_\ell}, o_\ell)$ be a Dynkin limit space. For any ℓ, Theorem A provides a function $f_\ell \in \mathcal{C}^2(M_\ell)$ with $0 \le f_\ell \le C(n, \gamma)$ such that the weighted Riemannian manifold $(M^n_\ell, \overline{g}_\ell := e^{2f_\ell} g_\ell, \bar{\nu}_\ell := e^{2f_\ell} \nu_{g_\ell})$ satisfies the $\mathrm{RCD}(K(n,\gamma)/T, N(n,\gamma))$ condition. The space of pointed $\mathrm{RCD}(K(n,\gamma)/T, N(n,\gamma))$ spaces is compact in the pointed measured Gromov–Hausdorff topology, hence we can assume that, up to extracting a subsequence, the sequence $\{(M^n_\ell, \mathsf{d}_{\overline{g}_\ell}, \bar{\mu}_\ell := c_\ell \bar{\nu}_\ell, o_\ell)\}$ converges to some $\mathrm{RCD}(K(n,\gamma)/T, N(n,\gamma))$ space $(X, \bar{\mathsf{d}}, \bar{\mu}, o)$. By [MN19, Theorem 1.1] and [BS20, Theorem 0.1] there exists $k \in \{0, \ldots, \lfloor N \rfloor\}$ such that $(X, \bar{\mathsf{d}}, \bar{\mu})$ is k-rectifiable. But $\mathsf{d} \le \bar{\mathsf{d}} \le e^{C(n,\gamma)} \mathsf{d}$ and $\mu \le \bar{\mu} \le e^{2C(n,\gamma)} \mu$, hence (X, d, μ) is k-rectifiable too. □

Remark 5.5. About the Mosco convergence of the Energy forms. In the setting of the proof of Proposition 5.4, the Dirichlet forms $\mathcal{E}_\ell$ and $\bar{\mathcal{E}}_\ell$ defined for any ℓ by

$$\mathcal{E}_\ell(u) := \int_{M_\ell} |du|^2_{g_\ell} d\mu_\ell, \qquad \bar{\mathcal{E}}_\ell(u) := \int_{M_\ell} |du|^2_{\overline{g}_\ell} d\bar{\mu}_\ell,$$

coincide. We know that the pointed measured Gromov-Hausdorff convergence of RCD spaces implies the Mosco convergence of the Cheeger energy, hence we get that the sequence $\{(M_\ell, \mathsf{d}_{g_\ell}, \mu_\ell, \mathcal{E}_\ell, o_\ell)\}_\ell$ converges in the pointed Mosco–Gromov–Hausdorff topology to $(X, \mathsf{d}, \mu, \mathcal{E}, o)$ where $\mathcal{E}$ is the Cheeger energy of $(X, \bar{\mathsf{d}}, \bar{\mu})$. This does not implies, a priori, that in the setting of Definition 5.1, the pointed measured Gromov–Hausdorff convergence self-improves to a Mosco convergence of the energy. Indeed, several choices of functions f_ℓ can be made and the limit distance $\bar{\mathsf{d}}$ and limit measure $\bar{\mu}$ could depend on the subsequence.

5.2. **Kato limit spaces.** Let $T > 0$ and $f\colon (0, T] \to \mathbb{R}_+$ be a non-decreasing function satisfying

$$f(T) < \frac{1}{n-2} \quad \text{and} \quad \lim_{t \to 0+} f(t) = 0.$$

Definition 5.6. A renormalized limit space $(X, \mathsf{d}, \mu, o) \leftarrow (M^n_\ell, g_\ell, \mu_\ell, o_\ell)$ is called a Kato limit space associated to f if $\mathrm{k}_t(M^n_\ell, g_\ell) \le f(t)$ for any ℓ and $t \in (0, T]$.

Proposition 5.4 together with [CMT22, Theorem 4.4] implies the following:

Proposition 5.7. Let $(X, \mathsf{d}, \mu, o) \leftarrow (M^n_\ell, g_\ell, \mu_\ell, o_\ell)$ be a Kato limit space. Then there exists $k \in \{0, \ldots, n\}$ such that (X, d, μ) is k-rectifiable, and for μ-almost every $x \in X$ the space $(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, \mathcal{H}^k, 0)$ is the unique metric measure tangent cone of (X, d, μ) at x.

5.3. **Non-collapsed strong Kato limit spaces.** Let $T > 0$ and $f\colon (0, T] \to \mathbb{R}_+$ be a non-decreasing function satisfying (SK). Note that the integrability condition from (SK) implies that $\lim_{t \to 0+} f(t) = 0$.

Definition 5.8. A pointed metric space (X, d, o) is called a non-collapsed strong Kato limit space associated to f if there exists a sequence of pointed complete Riemannian manifolds $\{(M^n_\ell, g_\ell, o_\ell)\}$ such that :

(1) $\mathrm{k}_t(M^n_\ell, g_\ell) \le f(t)$ for any ℓ and $t \in (0, T]$,
(2) there exists $v > 0$ such that $\nu_{g_\ell}(B_{\sqrt{T}}(o_\ell)) \ge v$ for all ℓ,
(3) $(M^n_\ell, g_\ell, o_\ell) \to (X, \mathsf{d}, o)$ in the pointed Gromov–Hausdorff topology.

Theorem C has the following consequence.

Proposition 5.9. Let $(X, \mathsf{d}, o) \leftarrow (M^n_\ell, g_\ell, o_\ell)$ be a non-collapsed strong Kato limit space. Then for any $x \in X$, the volume density

$$\uptheta_X(x) = \lim_{r \downarrow 0} \frac{\mathcal{H}^n(B_r(x))}{\omega_n r^n}$$

is well-defined. Moreover, there exists a function $c : (0, +\infty) \to (0, +\infty)$ depending on n and f only such that for any $x \in X$,

$$c(\mathsf{d}(o, x))v \le \uptheta_X(x) \le 1. \tag{16}$$

Proof. Because of the existence of cut-off functions established in Proposition 4.2, the adaptation of the proof of Colding's volume convergence theorem [Col97, Che01] done in [CMT21] carries over to the complete setting. As a consequence, one has $(M_\ell, g_\ell, \nu_{g_\ell}, o_\ell) \to (X, \mathsf{d}, \mathcal{H}^n, o)$ in the pointed measured Gromov-Hausdorff topology. Therefore, the conclusion of Theorem C passes to the limit and yields that, for any $x \in X$, $R \in (0, \sqrt{T}]$, $\eta \in (0, 1 - 1/\sqrt{2})$ and $r \le (1 - \eta)R$,

$$\frac{\mathcal{H}^n(B_R(x))}{\omega_n R^n} \exp\left(-\frac{C(n)\Phi(R)}{\eta}\right) \le \frac{\mathcal{H}^n(B_r(x))}{\omega_n r^n} \exp\left(-\frac{C(n)\Phi(r)}{\eta}\right).$$

Taking successively the limit inferior as $r \downarrow 0$ and then the limit superior as $R \downarrow 0$, we obtain that $\theta_X(x)$ is well-defined. As for (16), the lower bound is obtained as in [CMT21, Remark 2.18] and the upper bound as in [CMT21, Corollary 5.13]. □

In [CMT22], we proved that there exists $\delta = \delta(n, f) \in (0, 1)$ such that for any non-collapsed strong Kato limit space X associated to f, the dense open subset

$$\{x \in X : \theta_X(x) > 1 - \delta\}$$

is a topological manifold with Hölder regularity. This result was a consequence of the intrinsic Reifenberg theorem of Cheeger and Colding [CC97, Theorem A.1.1] applied to balls $B_{\sqrt{\tau}}(x)$ where the heat ratio is almost 1:

$$(4\pi\tau)^{\frac{n}{2}} H(\tau, x, x) \simeq 1.$$

In order to apply Cheeger and Colding's theorem, we showed a Reifenberg property for these balls by using two key results:

- the almost monotonicity of the heat ratio (that is to say Theorem 4.4 for the special value $s = t/2 = \tau/4$);
- a rigidity result for $\mathrm{RCD}(0, n)$ spaces $(Z, d_Z, \mathcal{H}^n)$ for which there exist $\tau > 0$ and $z \in Z$ such that $(4\pi\tau)^{\frac{n}{2}} H(\tau, z, z) = 1$ (see [CMT22, End of Proof of Theorem 5.9]).

Similarly, the almost monotonicity of the volume ratio granted by Theorem C and the rigidity of $\mathrm{RCD}(0, n)$ spaces with maximal volume ratio [DPG18, Theorem 1.1] naturally lead to the following result for balls with almost maximal volume:

Theorem 5.10. Let (X, d, o) be a non-collapsed strong Kato limit space associated to f. Then for any $\epsilon \in (0, 1)$ there exists $\delta \in (0, 1)$ depending only on n, f, ϵ such that if $x \in X$ and $r \in (0, \delta\sqrt{T}]$ satisfy

$$\frac{\mathcal{H}^n\left(B_r(x)\right)}{\omega_n r^n} \ge 1 - \delta$$

then the ball $B_{r/2}(x)$ satisfies the Reifenberg property, that is, for any $y \in B_{r/2}(x)$ and $s \in (0, r/4)$,

$$\mathsf{d}_{GH}\left(B_s(y), \mathbb{B}^n_s\right) \leq \epsilon\, s.$$

As in [CMT22], there are two alternative ways for deriving Hölder regularity of balls with the Reifenberg property: either by the intrinsic Reifenberg theorem of Cheeger and Colding mentioned above, or as a consequence of a recent idea of Cheeger, Jiang and Naber [CJN21] based on a transformation theorem. The advantage of the latter approach is that it gives a more quantitative statement. In [CMT22, Theorem 5.14] we proved a transformation theorem under a strong Kato bound, which can be easily rephrased for complete manifolds and under our weaker condition (SK) on the function f. Then the same argument as in [CJN21] yields the following :

Theorem 5.11. Let $f\colon (0,T] \to \mathbb{R}_+$ be a non-decreasing function satisfying

$$f(T) < \frac{1}{n-2} \text{ and } \int_0^T \frac{f(t)}{t}\,\mathrm{d}t < \infty.$$

For any $\alpha \in (0,1)$ there exists $\delta \in (0,1)$ depending only on n, f, α such that if (M^n, g) is a complete Riemannian manifold satisfying (K), for any $x \in M$ and $r \in (0, \sqrt{T})$ for which

$$\mathrm{k}_{r^2}(M^n, g) < \delta \tag{17}$$

and there exists a harmonic map $h\colon B_r(x) \to \mathbb{R}^n$ with $h(x) = 0$ and

$$r^2 \fint_{B_r(x)} |\nabla^g dh|^2\,\mathrm{d}\nu_g + \fint_{B_r(x)} \left|dh^t dh - \mathrm{I}_n\right|\mathrm{d}\nu_g \leq \delta, \tag{18}$$

then

i) $h\colon B_{r/2}(x) \to \mathbb{R}^n$ is a diffeomorphism onto its image;
ii) $\mathbb{B}^n_{\alpha r/2} \subset h\left(B_{r/2}(x)\right)$;
iii) $\forall y, z \in B_{r/2}(x)\colon \alpha r^{1-\frac{1}{\alpha}}\,\mathsf{d}_g^{\frac{1}{\alpha}}(y,z) \leq \|h(y) - h(z)\| \leq \alpha^{-1}\,\mathsf{d}_g(y,z)$.

Remarks 5.12. We conclude with some remarks on the previous statement.

- Following [CC00, Theorem 1.2] or [CMT23, Theorem A.1], it can be shown that the existence of a harmonic map satisfying (18) implies that

$$\nu_g\left(B_r(x)\right) \geq \left((1 - C(n)\sqrt{\delta}\right)\omega_n r^n.$$

- The condition $\mathrm{k}_{r^2}(M^n, g) < \delta$ is satisfied for r small enough, that is if

$$r \leq \sqrt{T} \exp\left(-\int_0^T \frac{f(s)}{\delta s}\,\mathrm{d}s\right).$$

- It can be shown that if f is as Theorem 5.11 , then for any $\delta \in (0,1)$ there exists $\eta \in (0,1)$ depending only on n, f, δ such that if (M^n, g) is a complete Riemannian manifold such that for all $t \in (0,T]$ and for some $r \in (0, \sqrt{T}]$

$$\mathrm{k}_t(M^n, g) \leq f(t)\ ,\ \mathrm{k}_{r^2}(M^n, g) < \eta \text{ and } \frac{\mathcal{H}^n\left(B_{2r}(x)\right)}{\omega_n (2r)^n} \geq 1 - \eta$$

then there exists a harmonic map $h\colon B_r(x) \to \mathbb{R}^n$ satisfying (18) (compare with [CMT21, Corollary 5.13]).

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