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The Social Epistemology of Mathematical Proof

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Abstract

If we want to understand why mathematical knowledge is extraordinarily reliable we need to consider both the nature of mathematical arguments and mathematical practice as a social practice. Mathematical knowledge is extraordinarily reliable because arguments in mathematics take the form of deductive mathematical ***proofs***. Deductive mathematical proofs are surveyable in the sense that they can be checked step by step by different experts, and a purported proof is only accepted as a proof by the mathematical community once a number of experts have checked the proof. Hence, the reliability of the body of mathematical knowledge is in part obtained through the surveyability of proofs and the ***social*** process of proof validation. This chapter reviews work relating to 1) the norm in the mathematical community of only counting as argument deductive mathematical proof, and 2) the norm of only counting as deductive mathematical proof an argument that has been confirmed to be a deductive mathematical proof by a number of experts. The chapter also presents cases of unusual mathematical proofs or arguments that challenge these norms.

Keywords

Social epistemology of mathematical proof; norms in mathematics; surveyability of proofs; proof validation; ghost theorems; complex proofs

1. Introduction

The aim of this chapter is to provide an overview of work relating to what we may term the social epistemology of proof. Not much work has been done on the social epistemology of proof but much work has been done that is highly relevant to the social epistemology of proof. The purpose of this chapter is to review this literature. The literature belongs to the philosophy of mathematical practice and I hope to create connections between parts of the literature in the philosophy of mathematical practice that have not been connected before. Furthermore, I hope to promote social epistemology of mathematics as an interesting area of research.

If we want to understand why mathematical knowledge is extraordinarily reliable we need to consider both the nature of mathematical arguments and mathematical practice as a social practice. Mathematical knowledge is extraordinarily reliable because arguments in mathematics take the form of deductive mathematical ***proofs***. Deductive mathematical proofs are surveyable in the sense that they can be checked step by step by different experts, and a purported proof is only accepted as a proof by the mathematical community once a number of experts have checked the proof. Hence, the reliability of the body of mathematical knowledge is in part obtained through the surveyability of proofs and the ***social*** process of proof validation.

This chapter has two main parts. The first part of the chapter reviews work relating to the norm in the mathematical community of only counting as argument deductive mathematical proof. The second part of the chapter reviews work relating to the norm of only counting as deductive mathematical proof an argument that has been confirmed to be a deductive mathematical proof by a number of experts. The chapter also presents cases of unusual mathematical proofs or arguments that either challenge the norm of only counting as argument deductive mathematical proof or the norm of only counting as deductive mathematical proof an argument that has been confirmed to be a proof by different experts.

2. The Community Value of Justification as Deductive Mathematical Proof

Many mathematicians have emphasized the major role of ***inductive*** reasoning in mathematics (see, for example, Borwein and Bailey 2003; Epstein and Levy 1995; Sørensen 2016). Mathematical results are typically discovered and tested in reliable ways before they are proved. To give an extreme example, the Riemann hypothesis has been verified in billions of instances. But there is really only one method of justification in mathematics and this is deductive mathematical proof. Mathematical results are established by proofs. And the Riemann hypothesis will remain a hypothesis until a deductive mathematical proof has been produced.

One way to identify the value of deductive proof is to ask why mathematicians are unwilling to accept other kinds of mathematical justification. We may wonder why the mathematical community is unwilling to accept proofs that make use of inductive evidence, even when the inductive evidence is extremely reliable, while the mathematical community does accept proofs that skip steps and are very long and complicated. I call the former randomized proofs in line with Jackson (2009). A proof that uses the Miller–Rabin primality test to establish that a specific number n is prime would be an example of a randomized proof that makes use of extremely reliable inductive evidence.

Philosophers have defended different views on whether the unwillingness of the mathematical community to accept randomized proofs is epistemically rational. Don Fallis (1997, 2000, 2002) argues that mathematicians' unwillingness to accept randomized proofs cannot be defended on epistemic grounds.

In response to Fallis, Kenny Easwaran (2009) offers an account of why mathematicians are unwilling to accept proofs that make use of inductive evidence. Easwaran claims that a randomized proof, as opposed to a traditional proof, is not transferable. By 'transferable' he means that the author can give the proof to other mathematicians who can then verify that the claim established by the proof is true just by considering the steps in the proof. They do not have to rely on the testimony of the author of the proof. In the case of a proof that uses the Miller–Rabin primality test to establish that a specific number n is prime, by contrast, the reader would have to rely on the author to have performed the Miller–Rabin test correctly. Easwaran suggests that this explains why mathematicians are unwilling to accept randomized proofs. Without going into much detail, Easwaran (2009, 361) states that requiring proofs to be transferable is a way for the mathematical community to protect the quality of the body of mathematical knowledge. In this way all the evidence for the knowledge is publicly available and can be independently checked by mathematicians with the relevant area of expertise.

Note that the fact that mathematical proofs can be independently checked by mathematicians separates mathematical proofs from scientific arguments. No matter how many scientists have read and become convinced by a paper arguing that p from empirical data, it cannot be excluded that the author of the paper has lied about having established p or has made a subtle mistake when collecting the data. Scientists can limit their dependence on the author by reproducing the experiments, but this is often not an option in practice. Reproducing experiments is expensive and time-consuming and generally does not give much credit.

Fallis (2011) agrees with Easwaran that the epistemic autonomy provided by transferable proofs can be epistemically valuable. However, he argues that randomized proofs also allow mathematicians to be epistemically autonomous. Given the proof from the example above, mathematicians can just perform the Miller–Rabin test for themselves to check that n is indeed prime. It is unclear to Fallis, as it is to me, why it is important that mathematicians who independently perform the Miller–Rabin test to establish that n is prime will not have the exact same evidence (2011, 172; Jackson 2009 gives a similar argument).

While Fallis and Easwaran disagree on what kinds of proof allow individual mathematicians to be epistemically autonomous, both Fallis and Easwaran emphasize that some level of autonomy of individual mathematicians is valuable to the mathematical ***community*** as a whole. Fallis (2011, 166–172) examines how some level of epistemic autonomy of individual mathematicians can be valuable to both the mathematical community as a whole and the individual mathematicians. It can be valuable to the mathematical community as a whole, he notes, because the community is more reliable when each proof is checked by more than one expert. Similarly, Easwaran calls transferability a "social epistemic virtue" presumably because protecting the quality of the body of mathematical knowledge is valuable to the mathematical community rather than any individual mathematician (Easwaran 2009, 343).

Don Berry (2016) also examines “mathematicians’ strict insistence on proof.” In line with Easwaran, Berry argues that there are good epistemic reasons for the mathematical community for only counting as argument deductive mathematical proof. Berry argues that there are some important virtues of mathematical research that are largely due to the norm in the mathematical community of only counting as argument deductive mathematical proof. And the success of mathematics, he further argues, is largely due to these virtues of mathematical research. Therefore, mathematicians are rational in their insistence on proof.

For example, Berry argues that the success of mathematics is partly due to the ***permanence*** of mathematical results. And the permanence of mathematical results, he argues, is a virtue that is due to the norm in the mathematical community of only counting as argument deductive mathematical proof. Like Easwaran and Fallis, Berry emphasizes as a virtue the possibility of ***autonomy*** for individual mathematicians when they read and assess a mathematical argument. For example, “When journal referees judge that a printed argument constitutes a sufficient basis for publication, they never rely upon trust in the testimony of the publishing mathematician” (Berry 2016, 117). This is due to the insistence on proof for “if a result were supported by inductive evidence alone, we would be required to accept the judgement of the author that this evidence is conclusive” (Berry 2016, 117). Berry emphasizes that the virtue of the possibility of autonomy for individual mathematicians helps maintain the virtue of permanence and other virtues.

To sum up, Easwaran, Fallis, and Berry agree that the norm of only counting as argument deductive mathematical proof allows individual experts to independently check mathematical arguments. And the fact that individual experts can independently check mathematical arguments is valuable to the mathematical community because it means that mathematical results can be extraordinarily reliable. This seems right. But the question remains as to how errors in a proof may be overlooked by individual mathematicians such that it makes a difference when more mathematicians check the same proof. Section 4 below addresses this question.

3. Challenging Cases I

It is worth briefly considering a phenomenon that seems to challenge the norm of only counting as argument deductive mathematical proof. I have in mind the phenomenon of ghost theorems. The notion of “ghost theorems” was introduced by Colin Rittberg, Fenner Tanswell, and Jean Paul Van Bendegem (2018, 6) and refers to “mathematical results which are taken as accepted in a mathematical community, relied upon in talks, discussion, and proving further results, but which cannot be traced to a concrete proof in the literature.” It remains to be studied how common ghost theorems are. Furthermore, it is unclear to what extent ghost theorems have unpublished proofs, but this is less important for our purposes because there is a substantial difference between unpublished proofs and published proofs. Rittberg et al. (2018, 8–10) emphasize that any unpublished proofs of ghost theorems are, all else being equal, less reliable than published proofs because they have not gone through the social process of proof checking that published proofs go through.

4. Proof Checking as a Social Process

A mathematical proof has traditionally been conceived as something very different from an ***argument*** as conceived in argumentation theory. In a key work in rhetoric, Chaïm Perelman and Lucie Olbrechts-Tyteca (1969) distinguish arguments from demonstrations and include mathematical proofs in the category of demonstrations. Demonstrations are impersonal as opposed to arguments which are relative to their intended audience. Perelman and Olbrechts-Tyteca (1969, 19) write that, "Since argumentation aims at securing the adherence of those to whom it is addressed, it is, in its entirety, relative to the audience to be influenced." In the same vein, Ralph Johnson writes that, "No mathematical proof had or needs to have a dialectical tier" (Johnson 2000, 270; quoted in Dufour 2013, 65). The dialectical tier of an argument is where the arguer anticipates and responds to objections to the argument.

While mathematical proofs have traditionally not been conceived as arguments this is changing as philosophers of mathematics have started to pay more attention to ordinary mathematical proofs as they occur in actual mathematical practice. For example, Erik Krabbe (2008/2013, 193) holds a "dialectical and rhetorical view of proofs" on which "proofs may be more or less successful, depending upon context and audience." Krabbe notes that, "Some proofs are rather sketchy [relative to the intended audience], leaving much room for questioning (or work for the reader, if the author is not there to answer questions), whereas others, by answering these questions, are developed to great dialectical depth" (2008/2013, 193). Other studies similarly emphasize the audience-dependence of proofs and point out that the appropriate level of detail of proofs depends on their intended audience (see, for example, Andersen, Johansen, and Sørensen 2019; Fallis 2003; Paseau 2016).

Ordinary mathematical proofs seem to be relative to their intended audience in at least some minimal sense. As mathematician William Thurston (1994, 175) has pointed out, mathematicians "prove things in a certain context and address them to a certain audience." It seems fair to say that a proof aims to secure the adherence of the audience to whom it is addressed, since a purported proof is accepted as a proof by the mathematical community only when enough experts have read and assessed the purported proof and concluded that it, in fact, constitutes a proof. Hence, philosophers of mathematical practice have argued that for proofs to be thoroughly checked and accepted as proofs by mathematicians they need to go through a ***social*** process of verification in which the proofs are read by different experts and used in different contexts (see, for example, Rav 1999; Devlin 2003; and Andersen, Andersen, and Sørensen 2020).

Not until a number of experts have checked a proof does the proof count as a proof in mathematical practice. This is so because individual readers of proofs may quite easily overlook errors (see, for example, Geist, Löwe, and Van Kerkhove 2010; Müller-Hill 2011).[1] Mathematicians themselves stress that errors in proofs may quite easily be overlooked. Philosopher Eva Müller-Hill has interviewed six mathematicians about their intuitions as to when a mathematician may be said to know that a mathematical claim is

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[1] This may be one reason why it is becoming more common, although still rare, to formalize proofs as a way of verifying them (Avigad 2008; Avigad and Harrison 2014; Hales 2008). This kind of proof checking is not in general a social process but the formalization of a proof will sometimes be the result of a collaborative effort.

true. Five of them stressed that even an expert journal referee may overlook errors (Müller-Hill 2011, 305–310, 315–316, 319–322, 328, 342–343). In the words of one of the interviewees, pre-publication peer review is just "one step in trying to make sure that nobody has overlooked a mistake in the proof" (Müller-Hill 2011, 309). That errors may quite easily be overlooked by any individual mathematician is also reflected in the fact that mathematicians seem to be hesitant to rely on a result without checking the proof of the result themselves unless a number of other experts have checked the proof (Andersen, Andersen, and Sørensen 2020).

Because the vetting of a proof is a long and complex process and errors in proofs are sometimes only found after publication, some have argued that the mathematical community should have a better system for sharing publicly the errors found in published proofs (Grcar 2013; Frans and Kosolosky 2014).

I will now review a number of different accounts of proof since different accounts of proof come with somewhat different accounts of why it makes sense to have more experts check a proof. This is due to the fact that different accounts of proof come with different accounts of how individual mathematicians read and assess proofs. And, therefore, different accounts of proof come with different accounts of how errors in a proof may be overlooked by individual mathematicians.

A number of philosophers claim that ordinary proofs are what one could call ***sketches*** of formal proofs (see, for example, Avigad 2020; Azzouni 2004; Hamami 2019; Mac Lane 1986). A purported proof is a proof if it can be turned into a formal proof. If we conceive of proofs this way, then becoming convinced that a purported proof is a proof involves recognizing that the purported proof could be turned into a formal proof. Jeremy Avigad describes the process thus,

> Whether or not a mathematician reading a proof would characterize the state of affairs in these terms, a judgement as to correctness is tantamount to a judgment as to the existence of a formal derivation, and whatever psychological processes the mathematician brings to bear, they are reliable insofar as they track the correspondence. (Avigad 2020, 3)

On the proofs as sketches accounts, overlooking an error in a proof amounts to making a mistake in tracking the correspondence between the ordinary proof and the formal proof. Some philosophers have criticized the proof as sketch account (see, for example, Larvor 2012; Rav 1999; Tanswell 2015). The discussion is ongoing.

Others argue that we should conceive of proofs as ***narratives*** (Doxiadis 2012; Robinson 1991, 269; Thomas 2007). For example, we may conceive of a proof as a telling of how the author of the proof came from the premises to the conclusion, as a telling of a sequence of inferential actions performed by the author (Doxiadis 2012). This fits well with how mathematicians have occasionally described proving as taking a journey (as described in Lane et al. 2019, 199–200). If we conceive of proofs as narratives in this way, then coming to the conclusion that a purported proof is a proof involves an assessment of the actions performed by the author of the proof. Overlooking an error in a proof amounts to something like making a mistake when assessing whether a certain action is suitable for obtaining a certain goal.

Other accounts of proof are similar to the proof as narrative account in that they focus on actions, but they focus on the actions of the readers of proofs rather than the authors of proofs. Hence, some philosophers argue that we should conceive of proofs as ***recipes*** for how to prove propositions (Larvor 2012, 725–726; Sundholm 2012; and Tanswell

2017). On these accounts, a proof as written is a recipe for the readers to follow to do an actual proof. If proofs are recipes, then reading and assessing a purported proof involves actually performing the steps in the recipe and seeing if they lead to the desired outcome. In this case, the reader is doing the proving herself under guidance, or taking the journey herself. If we conceive of proofs as recipes, then coming to the conclusion that a purported proof is a proof amounts to performing the inferential actions oneself. Overlooking an error in a proof amounts to making a mistake when doing the proof.

Finally, some philosophers have argued that we should conceive of proofs as ***dialogues*** (Dutilh Novaes 2016, 2018; Ernest 1994). Catarina Dutilh Novaes gives an account of a proof as a dialogue between two fictitious interlocutors whom she calls Prover and Skeptic. On her account,

> A deductive proof corresponds to a dialogue between the person wishing to establish the conclusion […] and an interlocutor who will not be easily convinced and will bring up objections, counterexamples, and requests for further clarifcation. A good proof is one that convinces a fair but 'tough' opponent. (Dutilh Novaes 2016, 2617).

She adds: "Ultimately, most of the work is done by Prover, but Skeptic has an important role to play, namely to ensure that the proof is persuasive, perspicuous, and valid" (Dutilh Novaes 2016, 2618). If proofs are dialogues in this sense, then reading and assessing a proof involves assessing whether Skeptic is skeptic enough. Overlooking an error in a proof often amounts to overestimating the skepticism of Skeptic. Note that the accounts of proofs as dialogues are perhaps especially interesting from the point of view of social epistemology because they include the social nature of dialogue in the very nature of proof.

Conceiving of proofs ***as*** dialogues should be distinguished from conceiving of proofs as the ***result of*** dialogues, as Lakatos (1976) famously did. In *Proofs and Refutations*, Lakatos illustrates how mathematical concepts can grow through a dialectic of proofs and counterexamples, using the historical example of the concept of polyhedron as used in proofs of Euler's conjecture. The concept of polyhedron becomes more clear and precise in the context of a long debate of the proofs, which are also refined in the process. The book is structured as a dialogue between a teacher and his students, with the students representing different methodological stances (see Larvor 1998, 10–11).

5. Challenging Cases II

The different accounts of proofs we have considered provide different answers to the question of why a purported proof is only accepted as a proof by the mathematical community when enough experts have read and assessed the purported proof and found no errors. That is, when the purported proof has gone through a social process of proof validation. But there are some unusual cases of proofs that challenge the validation of proofs as a social practice. More specifically the cases challenge the norm of only counting as deductive mathematical proof an argument that has convinced a number of experts that it is indeed a deductive mathematical proof.

A number of prominent and highly praised proofs have received attention for being so long or complex that only a few mathematicians have read and checked the proofs. Famous examples of proofs that have received attention for being so long that they would be incredibly hard for any individual mathematician to read include the proof of the classification of finite simple groups (see, for example, Steingart 2012) and the proof of the Kepler Conjecture by Thomas Hales (see, for example, Devlin 2003). The famous proof of Fermat's Last Theorem by Andrew Wiles is an example of a proof that draws on different domains in such a way that only few individual experts have been able to read the proof (see, for example, Jones and Jones 1998, 217–237; Penrose 1997). These proofs are in practice not very open to checking by mathematicians.[2] For this reason, they pose a challenge to the norm of only counting as deductive mathematical proof an argument that has been checked by a number of experts.

These are proofs of very high-profile results that ***challenge*** the norm. Very long or complex purported proofs of less important results presumably often ***succumb*** to the norm in the sense that they are not accepted as proofs by the mathematical community because they have not been vetted by enough experts. The norm of only counting as deductive mathematical proof an argument that has been checked by a number of experts limits the extent to which purported proofs can be long and complex and still be accepted as proofs by the mathematical community.

But even proofs of high-profile results by high-profile mathematicians may succumb to the norm of only counting as deductive mathematical proof an argument that has been checked by a number of experts. The purported proof of the abc conjecture by the prominent mathematician Shinichi Mochizuki has been described as being extremely hard to even begin to comprehend (see Castelvecchi 2015 and Hartnett 2014). The

[2] Note that this means that extremely few mathematicians could rely on these proofs in their work and still be autonomous.

purported proof is from 2012 and has not yet been accepted as a proof by the mathematical community and perhaps will not be. Mochizuki had been working on the proof for many years on his own and had in that time "constructed his own mathematical universe and populated it with arcane terms like 'inter-universal Teichmüller theory' and 'alien arithmetic holomorphic structures'" (Hartnett 2014, 225). Hence, the purported proof is written in a new language and, in addition, the author is rather unwilling to explain his thinking. The behavior of the author can perhaps be described as going against the norm of the mathematical community of only accepting as justification in mathematics arguments that are open to independent checking by different mathematicians.

6. Conclusion

If we want to explain the extraordinary reliability of mathematical knowledge we may point to the central role of deductive mathematical proofs in mathematics. But to understand the reliability of mathematical proofs we also need to consider the relationship between mathematical proofs and mathematical practice as a social practice. We need to consider the social epistemology of mathematical proof. In this chapter I have reviewed how philosophers have examined how mathematical proofs are valuable to the mathematical community because of their being open to independent checking by different mathematicians. I have also reviewed how philosophers have offered different accounts of how mathematicians read and assess proofs. These accounts offer different answers to the question of how individual mathematicians may overlook errors and thus of why we need a social process of proof checking.

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