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Mathematicians Writing for Mathematicians

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Abstract

We present a case study of how mathematicians write for mathematicians. We have conducted interviews with two research mathematicians, the talented PhD student Adam and his experienced supervisor Thomas, about a research paper they wrote together. Over the course of two years, Adam and Thomas revised Adam's very detailed first draft. At the beginning of this collaboration, Adam was very knowledgeable about the subject of the paper and had good presentational skills but, as a new PhD student, did not yet have experience writing research papers for mathematicians. Thus, one main purpose of revising the paper was to make it take into account the intended audience. For this reason, the changes made to the initial draft and the authors' purpose in making them provide a window for viewing how mathematicians write for mathematicians. We examined how their paper attracts the interest of the reader and prepares their proofs for validation by the reader. Among other findings, we found that their paper prepares the proofs for two types of validation that the reader can easily switch between.

Keywords

Mathematical publication; mathematical audience; mathematical argument; the nature of proof; contextualization

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1. Introduction

Traditionally, mathematical proofs have been conceived of as formal derivations from a given set of axioms (e.g. Hilbert 1925/1967). On this view, proofs are radically different from *arguments* as they are conceived in argumentation theory. Thus, in a foundational work in argumentation theory, Chaïm Perelman and Lucie Olbrechts-Tyteca (1969) separate arguments from demonstrations, such as mathematical proofs. Demonstrations are impersonal, arguments are not; the authors emphasize that, "since argumentation aims at securing the adherence of those to whom it is addressed, it is, in its entirety, relative to the audience to be influenced" (1969, p. 19). On their account, whether the argumentation is cogent is determined by the audience.[1] Similarly, Ralph Johnson writes that, "no mathematical proof had or needs to have a dialectical tier" (2000, p. 270; quoted in Dufour 2013, p. 65). The dialectical tier of an argument is where the arguer anticipates and responds to objections to the argument. Mathematical proofs are not relative to the audience in this sense, but are, on this traditional account, objective in the radical sense that they are independent of audience and intellectual context.

And yet, this ideal is rarely met in actual mathematical practice. Encouraged by the ground-breaking work by Imre Lakatos (1974) and mathematician and philosopher Reuben Hersh, who wanted "a philosophy that is true to the reality of mathematical experience" (1979, p. 40), philosophers of mathematics have started to give ordinary mathematical proofs, as they occur in actual mathematical practice, more attention. The resulting work has shown that proofs as they are commonly found in mathematical practice often differ from formal derivations (e.g., Lakatos 1974; Rav 1999; Manders 2008). In fact, as the case of the computer-generated proof of Robbins' conjecture shows, even when a proof is presented to us as a fully formal derivation, common mathematical practice compels mathematicians to reformulate the proof and present it in less formal and more (humanly) accessible style (e.g. Dahn 1997).

[1] For an account of the relevance of this work to the philosophy of mathematics, see Dufour 2013.

This body of work indicates that ordinary mathematical proofs *are* relative to their audience in several ways. Especially, in ordinary mathematical proofs, some logical steps are typically omitted, and appeals to the intuitions and background knowledge of the reader are ubiquitous, if often tacit. Furthermore, mathematicians submit themselves to the interests and perceived argumentative values of the audience (Johansen and Misfeldt 2016). Consequently, what a proof looks like depends on the intended audience of the proof. As mathematician William Thurston has pointed out, mathematicians "prove things in a certain context and address them to a certain audience" (1994, p. 175). Hence, to understand the nature of mathematical proofs, we need to look at them in the context of their intended audience and examine how they are made to address it. We have to recognize that mathematics is a social practice and look at proofs in their social and communicative contexts.

This development in the philosophy of mathematics suggests that argumentation and communication theory may, after all, be relevant to the study of the nature of mathematical proof. Indeed, a recent collection of essays studies the relevance of argumentation theory in the philosophy of mathematics (Aberdein and Dove 2013). In one essay, for example, Erik Krabbe defends a "dialectical and rhetorical view of proofs" on which "proofs may be more or less successful, depending upon context and audience" (2013 [2008], p. 193; see also Krabbe 2013 [1997]). Krabbe notes that, "Some proofs are rather sketchy [relative to the audience in question], leaving much room for questioning (or work for the reader, if the author is not there to answer questions), whereas others, by answering these questions, are developed to great dialectical depth" (2013 [2008], p. 193). Similarly, studies that examine the level of granularity of ordinary proofs emphasizes the audience-dependence of proofs; they point out that the appropriate level of granularity of a proof depends on its intended audience (e.g., Fallis 2003; Paseau 2016; Andersen 2018). For example, the level of granularity of a textbook proof written for high school students will often be higher – meaning that more details are included – than the level of granularity of research proofs. However, these studies do not go into detail with *how* the level of granularity of a proof depends on the audience.

In this paper, we focus on research papers and arguments written for researchers in mathematics. Through a case study, we examine how the authors of a mathematical research paper use explanatory text to capture and retain the interest of the reader and how they prepare their proofs for validation by the reader. Among other things, the mathematicians in our case prepared their proofs for two types of validation that require different levels of granularity of the proofs: At one level, arguments are motivated to be read in outline, and at another level, details are provided to allow for a step-by-step reading. Andersen (2018) similarly suggests that research proofs are intended to present themselves to the reader at two levels of granularity. Where Andersen (2018) is mainly focused on the perspective of reviewers of mathematical research papers, the primary goal of the current case study is to examine the subject in more detail from the point of view of the authors of the papers.

2. Investigating the genesis of a mathematical paper: What we did

In order to investigate the authors' perspective on mathematical writing we contacted the experienced mathematician, Thomas, who gave us access to the drafts of a paper written by himself and his talented PhD student Adam. The first draft of the paper was written by Adam alone but during the next two years, Adam and Thomas conducted a series of revisions to the paper. At the beginning of this collaboration, Adam was the expert on the mathematics of the paper and already had good expositional skills but, as a new PhD student, he did not yet have experience in writing a research paper for a mathematical audience and did not yet know the expectations and preferences of such an audience. Thus, one main purpose of revising the paper was to make it take the intended audience into account. For this reason, the changes made to the initial draft provide a window to how mathematicians write for mathematicians and, in particular, how they frame their proofs when writing for mathematicians. In this sense, our case provides a quite unique way of exploring the topic of how mathematicians write for mathematicians.

In order to explore the particular perspective of the authors further, we conducted interviews with Adam and Thomas about the changes made to the manuscript. For practical reasons the interviews were conducted over e-mail. We sent the same sets of questions to both Adam and Thomas in joint e-mails, but asked them to respond to the questions individually, explaining that they represented two different perspectives – that of the PhD student and that of the experienced researcher – and we wanted to be able to separate the two perspectives. In this paper, we focus on Thomas' perspective, the perspective of the experienced mathematician. Follow-up questions were sent to them individually. We asked them to give answers that were as specific as possible and told them not to worry about whether their answers were in line with what they would usually do in similar situations. We described to them how each question would ask about a particular change they made to their paper and asked them to describe in detail what their thoughts were with respect to that particular change. Adam and Thomas were presented with the final manuscript of this paper for comments and qualification.

When developing the interview questions, we chose to compare only the first and last version of the paper by Adam and Thomas (by the 'last version' mean the version of the paper that was submitted to a journal and subsequently published in one of the leading journals in the specific field). We made this choice because we wanted to investigate the contrast between the writing of an experienced and an inexperienced mathematician. For this reason, we chose to focus on the changes that were important in the sense that they survived until the manuscript was submitted rather than investigating the back-and-forth revisions through the various versions of the manuscript.

In all of the questions we asked about the purpose of particular changes made to the paper. Based on our preliminary analyses, we decided to focus on the following four *types of action:*

1) adding or removing context (that is, adding or removing a reference or a description of the context; mainly mathematical but also historical context)

2) enhancing or lowering the level of detail (mainly mathematical but also historical detail)

3) adding or removing a narrative voice (see below)

4) adding or removing results.

In the interviews and in this paper, we use the term 'narrative' in a pre-theoretical sense without reference to narrative theory as such. We use the term to refer to three different aspects of Adam and Thomas' writing. First, when we speak about a narrative or *contextualizing* voice in their paper, we refer to a passage in which a story is told about how the results proved in the paper and related results have been studied before by other mathematicians. Second, when we speak about a narrative or *explanatory* voice in their paper, we refer to a passage in which a story is told about how the authors developed a certain proof or how the proof will progress in their paper. Finally, when we speak about the narrative of their paper, we refer to the *coherent story* told in the paper as a whole.

In addition to the four types of action, we decided to initially focus on three *purposes* that parts of the paper may have:

1) to convince the reader that the results are interesting

2) to convince the reader that the results are new

3) to convince the reader that the results are correct.

We were very open to the interviewees bringing up other and maybe less broad purposes. It is worth noting that the answers we received confirmed that the listed purposes are important. Our questions mainly focused on purpose 3) and to a lesser extent on purpose 1).

In the first two rounds of questions, the interview questions, except for the follow-up questions, roughly had the form:

> *In this part of the paper (e.g., this paragraph, sentence, or part of a sentence) you seem to aim to convince the reader that a result in the paper or the paper in general is {interesting, new, correct} by {adding, removing} {context, detail, narrative, results}. If this is accurate, please describe why you took this action to further this aim. If not, please describe what your aim was with the action.*

In the third and last round of questions, we refrained from using the expression 'to convince the reader' because Adam and Thomas rarely used the verb 'convince' in their answers. Instead, we used expressions such as 'to show the reader', 'to enable the reader to see', and the like. Furthermore, we included more specific aims in our questions than the three aims stated above.

Two methodological concerns are worth addressing in connection to the format of our interview questions. Firstly, we used 'why'-questions even though this type of questions may invite rationalizations. The aim of the interviews, however, was to investigate the perspectives of the authors on the changes they had made to their manuscript. In our view this perspective includes the authors' conception of norms and good practices in their academic field that may be revealed through rationalizations.

Secondly, since we were interviewing experts about their area of expertise, we were comfortable asking leading questions, at least as a way of beginning.[2] We observed that neither Adam nor Thomas seemed at all hesitant to correct us when we got their motivations wrong, and we therefore continued to ask leading questions. For example, when we

[2] We were asking leading questions in the sense of asking questions about the aims of the changes they made to the paper while suggesting what these aims may be. Lewis Anthony Dexter wrote that leading questions "are helpful in interviewing experts; most experts are predisposed to argue about professional matters and set people right, and few of them are so malleable as to fall *tout court* for a leading question" (1970/2006, p. 79).

suggested that three examples were added to the paper to faster convince the reader that certain results were correct, Thomas begins his answer with, "The examples were each given for specific reasons, none of which, however, was to convince the reader of any results' correctness." In another case, where we also suggested that a certain change was made to faster convince the reader that a result was correct, Adam at length explained what he was motivated by, which was not at all what we had suggested. He ended his answer with, "I was not at all motivated by how the proof convinces the reader of the correctness of the theorem."

The interviews took place over a period of five months, with one break of three months in between the second and third round of questions. After receiving their answers to the second set of questions and follow-up questions, we had already been provided with a lot of data. We decided to make a preliminary coding and analysis of these data before continuing with the interviews. That way the answers we had already received could better guide the questions we asked next.

When we coded the interviews, we drew on a theoretically based understanding that different aspects of a mathematics paper may serve different epistemological purposes (Andersen 2018). This understanding and our data led us to make the following coding taxonomy based on a distinction between Level I and Level II of Adam and Thomas' research paper:

Level I: The parts of the paper *outside* the parts called Definition, Proposition (or something similar, like Lemma and Theorem), and Proof

Level II: The Definitions, Propositions, and Proofs

The final coding taxonomy comprises four main themes with one to four subthemes each. The main themes are: 1. *Additions* made at Level I of the research paper and their purpose (subthemes: a. describing and evaluating relevant results from the literature, and the purpose thereof; b. describing and evaluating relevant approaches from the literature, and the purpose thereof; c. adding examples as illustrations of definitions, results, or approaches, and the purpose thereof; d. adding textual structures to retract and retain the interest of the reader. 2. General comments on the purpose of the additions made at Level I (subthemes: a. making the proofs easier for the reader to understand and validate; b. making it clear to the reader why the results are interesting and new; c. creating the flow of the paper). 3. *Deletions* made at Level II of the research paper and their purpose (subtheme: a. "cleaning" the proofs by removing a narrative voice explaining how the proof will progress and deleting 'we's, and the purpose of this). 4. The authors' attitude towards the reader (subthemes: a. the authors' attitude towards the reader at Level I; b. the authors' attitude towards the reader at Level II).

In the next two sections, we tell the story that has emerged through the analysis of changes made to Adam and Thomas' paper and the analysis and coding of the interviews about their purpose with making the changes. When we contrasted the two versions of their paper, we observed that it had been revised such that the narrative or explanatory voice that explains why the authors proceed the way they do had been removed from proofs, and Level II content in general. Also, the authors did not contextualize the mathematical results by, e.g., referring to similar results from the literature in the Level II parts of the final version of the paper. By contrast, Level I had been changed so that it constitutes a coherent story. Furthermore, Level I parts of the paper contextualize the mathematical results by describing their history, among other things. The next two sections, Section 3 and 4, examine the purpose of the coherent story and contextualization at Level I, while Section 5 examines the purpose of the *lack* of explanatory voice and contextualization at Level II.

As a final methodological point, it should be noted that our study is idiographic and we cannot lay claim to wide generality. The empirical basis of our paper is a case study, where we investigate a single manuscript and the authors' unique perspectives on their work with this manuscript. This being said, the paper we investigated was accepted for publication by a very respected mathematics journal and Thomas is not just an experienced mathematician but also an editor for another journal in the field. The case study thus embodies norms and standards of a sub-part of the mathematical research community, and consequently the choices and considerations represented in our paper may reasonably be said to reveal something generic about the writing practice of (parts of) the mathematics community.

3. Adding Narrative to Make the Results More Attractive

In the last version of their paper, Adam and Thomas do different things at Level I to create a coherent story and to contextualize their results. For example, they describe and evaluate relevant results from the literature (see Figure 1 and 2); they describe relevant approaches from the literature; and they give examples to illustrate definitions, results, and approaches (their own and those of others). One may say that the coherent story at Level I is like a novel in that it tells a story about interrelated characters that we get to know and become familiar with by being told about their history and about how they behave in particular situations. But instead of people, the characters are mathematical concepts, results, and approaches.

In [?] Mubayi's conjecture is proven for the special case $d = k$ and a weaker result in general. The Conjecture may still hold even if $d > k$. Indeed Füredi in [?] proves the $d = k + 1$ case of Mubayi's conjecture. However in [?] Mubayi gives a counter example which shows the largest d for which Mubayi's conjecture can hold is less than 2^k (for fixed k).

Figure 1. Contextualization in the literature, first draft

As described above, Mubayi [15] proved the conjecture for $d = 3$, and Mubayi [16] proved the conjecture for $d = 4$ when n is sufficiently large. Füredi and Özkahya [11] and Mubayi and Ramadurai [17] independently improved this result by proving the conjecture for sufficiently large n, thus generalising the above-mentioned result by Frankl and Füredi [9]. Chen et al. [4] proved Mubayi's Conjecture for $d = k$ and Füredi and Özkahya [11] proved that Mubayi's Conjecture even holds when $d = k+1$. However, Mubayi [16] provided a counterexample that showed that the conjecture could not be extended to values of d greater than or equal to 2^k.

Figure 2. Contextualization in the literature, final draft

This and the following section are about the answers Adam and Thomas gave when asked about the purpose of revisions they made to Level I to make it provide a more coherent story and include more context. It examines two primary purposes of these revisions at Level I.

One primary purpose is to attract and maintain the attention of the reader by making clear to the reader that the results are interesting and important. For example, we posed the following question with reference to the first pages of the paper, more specifically with reference to pp. 1–2 in the initial version of the paper and pp. 1–3 in the last version of the paper,

> *One way in which you changed this part of the paper is by adding narrative and context. You have added a narrative voice that is evaluative when telling the reader about the "celebrated Erdös-Ko-Rado Theorem" and that tells the story of how the theorem has been extended (by Frankl, Frankl & Füredi, and Mubayi) and of how the theorem has been conjectured extended (by Katona and Mubayi). This voice gives context to your results.*
>
> *By adding this narrative voice that places your results in a context in this way, you seem to aim to better convince the reader that your results are interesting and new. If this is accurate, please describe why you took this action to further this aim. If not, please describe what your aim was with making the described change to the paper.*

Thomas responded,

> *[…] Some mathematical results might seem evidently useful or beautiful to a wide audience, but most mathematical results need context in order to be properly appreciated. Like art, mathematics shines best when context is given; and, like an obscure technological device, one can best appreciate mathematics when one understands why it might be useful.*
>
> *At an immediate and practical level, editors and referees of mathematical journals face a torrent of mathematical submissions that they must quickly sort through. So, as an author, I try to sell the paper's results well, by presenting them as quickly and clearly and with gripping and contextual narrative.*
>
> *This salesmanship and presentation is equally important after the paper might be published, since I also want to tempt readers to read the paper, be excited by its results, and to cite it or maybe even to initiate collaboration with us. […]*

Thomas went on to say that there are also other motivations behind the contextualization in question. He explained that the contextualization also helps them, the authors, see what they should include in or exclude from the paper, given that they want to tell a coherent story. It may help them see that some "results, though of possible individual value, might stand out too much and not be suitable to include in the paper."

Situating their work in the literature is not the only means by which Adam and Thomas attract and maintain the attention of the reader. In Thomas' words above the narrative should be "gripping". This goal was pursued not only by adding context but also by reworking the communicative aspects of the text. As illustration, we observed that the revised version of the paper contained a recurring template where 1) something is described, 2) a problem is pointed out and 3) because of that some action needs to be taken. This template is referred to as the 'and, but, therefore'-template (ABT) in the science communication literature (Olson 2015, p.16), and is a well-known tool used by screen writers and other narrators to make their stories more compelling and interesting.

We asked Thomas whether the introduction of this structural template in the final version of the paper was a conscious choice or something he had done intuitively based on his experience writing papers for his peers.

In his answer, Thomas told us that it had in fact been a bit of both. Concerning the first instance where the structure is used, Thomas wrote:

> *First, I used my experience, seeing that the contents could be presented by a narrative structure that I have often used and which seems natural and strong to me. After that initial choice, my dominant mode of thinking was to be explicit and conscious about the smaller details of the narrative, down to grammatical structures and individual phrases and words.*

In other words, the introduction of the ABT-template was not done haphazardly or as the result of a stroke of luck. Rather, it was introduced deliberately and with great care.[3]

We furthermore asked Thomas if the aim of adding the "narrative structure" was to show the reader that the results of the paper are interesting. Thomas in part confirmed this interpretation:

> *I do indeed try to show the reader that my results are interesting and I do so by way of narrative. However, I have two distinct and differing aims for doing so. The first is prosaic: I would like to seduce and impress referees and editors so that they will get my papers published. The second aim is to produce beautiful work that will please me and hopefully other readers. […O]ne could even say that many strong and deep research results are worthless without a strong and convincing narrative to give them life and purpose.*

These answers show that an important motivation for describing the need for a particular result is connected to the social nature of mathematical research. Where the PhD student, Adam, is mainly concerned with results themselves and the correctness of the proofs, the experienced supervisor, Thomas, is aware that mathematics is not a solitary game. He wants other mathematicians to read his work, and he deliberately writes in order to keep the reader engaged and to convince her that the results are important and worthwhile.

This observation corresponds well with results of a qualitative interview study conducted by Johansen and Misfeldt. Here, the wish to be read by other mathematicians was identified as a guiding principle for mathematicians in choosing research problems, and this principle was found to have both a strategic and an emotional component. In order to build a career, mathematicians need to be published and quoted, but when asked, the mathematicians also expressed their need to resonate with other mathematicians and to feel that they contribute to something bigger than themselves (Misfeldt & Johansen 2015; Johansen & Misfeldt 2016). In the comments made by Thomas we see the exact same two concerns; on the one hand he wants to get his paper published (strategic concern), but he also wants to be read, to find possible new collaborators and to create something beautiful others can recognize as such.

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[3] We should stress that we used the term 'narrative structure' in the question Thomas is responding to here. Since in using the term Thomas is merely copying our choice of words for the structural template in question, we ascribe no significance to his use of this term.

4. Adding Narrative to Make the Proofs Easier to Understand

Another primary purpose of the coherent story at Level I is to make the content at Level II – i.e. the content of the proofs, propositions, and so on – easier to understand. This purpose is the focus of this section. Thomas ends the answer quoted in part in the beginning of the previous section by saying that the coherent story or narrative of the paper and the context are also important because "they make the technical details of the paper much easier to digest and understand, making both details and overall ideas easier to visualize." We were curious as to what he meant by this and what the implications were for proof validation, so in a follow-up question we asked,

> *You write that the narrative and context help explain your overall ideas to the reader. Can understanding your overall ideas be of help to the reader when she wants to validate particular proofs in your paper? If yes, how?*

It is worth quoting Thomas' answer in full,

> *The proofs of the paper are fairly technical and involve technical definitions, including definitions of shifting, stability, stars and the several variants of conditionally intersecting conditions. Once the reader has a reasonably clear visualisation of these concepts, then the reader does not need to repeatedly and painstakingly refer to the definitions of these concepts when reading a proof, and a visualisation of the overall proof in question can begin to crystalise, giving a local narrative for the detail to fit naturally into. The general narrative [of the paper] can help create these visualisations. Indeed, even the terminology can help – including 'shift', 'stable' and 'star' – to lead the reader to quickly form a clear visualisation of the concepts, borrowing from the latent associations that these words normally carry.*
>
> *[…] Indeed, those aspects are then easy to visualise: the first part of the proof has the structure of an induction proof involving a splitting into cases, namely the splitting of the family into two smaller families. These are set analogues of deletion and contraction for graphs, or puncturing and shortening for codes, so here too the reader might find concepts easy to visualise, given possible associations to prior visualisations. One might argue that these operation pairs provide a mini-narrative in themselves, a recognisable pattern that the reader can use and find some affinity and even comfort with.*

Here, Thomas puts great emphasis on the purpose of enabling the reader to "visualize" concepts. In order to achieve this, examples and comparisons with familiar concepts are important. To "visualize" a concept seems to imply creating a mental representation of the concept.

Once the reader has visualized the central concepts of a proof, or gained familiarity with them, she can begin to "visualize" the structure[4] or the flow of the proof without knowing the details of the proof. Thus, she can begin to understand or follow the proof without studying the details, just by looking at the structure of the proof or the "pattern" of the proof (a term he uses later in a similar context, as we shall see): by somehow looking at the proof in broad outline. If the reader from prior experience has "visualizations" of or familiarity with the operations used in the proof, this can also help her understand the proof by looking at it in broad outline.

We have not yet addressed the question of how the reader is provided with visualizations of or familiarity with the relevant concepts. Thomas emphasizes that the story told at Level I can help with this. Everyday experiences can also help in so far as the concepts in question are related to everyday experiences, which he points out that 'shift', 'stable' and 'star' *are*. It is not clear from the quote how the content at Level I can help the reader visualize concepts, but it seems that illustrative examples of concepts play an important role in this. Another part of the interview provides an example. When asked about the purpose of including three specific examples in the paper, Thomas explained that,

> *Example 2.2 was to provide a simple and clear example of stability and, indirectly, shifting. The example also provided the opportunity to point out that stars were stable, thus priming the reader for the contents of Theorem 2.3, hopefully making that theorem and the role of stars slightly more accessible.*

To sum up, a reader's "visualizations" of the central concepts in a proof can give her "a visualization of the overall proof in question," or can make her capable of following the proof from looking at the proof in broad outline. We wanted to know whether this would enable the reader to *validate* the proof without checking all the details, so we asked Thomas,

> *When the reader has "a visualisation of the overall proof in question" does this sometimes enable her to check that the proof is correct without going through every detail of the proof?*

He said,

[4] From comparing the two paragraphs of the answer, it seems that Thomas uses the term 'local narrative' to refer to the structure of the proof. We should not put too much emphasis on his use of the word 'narrative' here, since he may just use it because we do, even if we do not use it in this particular sense.

Yes, definitely!

Or, more precisely: once the reader has this visualisation, they also then have a recognition of pattern or flavour of the proof. The verification is then no longer one of meticulously checking each detail for logical accuracy and to try to fit this detail into an overall narrative ("What was the purpose of this detail for the proof?"), a narrative that one is trying to form. Instead, with narrative or visualisation at hand, one verifies the proof to a greater degree by seeing whether the general pattern and, at a glance, the details, of the proof are recognisable in their pattern and flavour. If everything feels correct, and some additional checking of the more oblique or technical parts of the proof also gives an affirmative outcome, then the proof can be said to be true. However, if some of this pattern recognition raises a red flag or indeed gives any hint of unease or alienation, then there might be flaws or errors, and the verification process should then revert to checking details at a sub-narrative level.[5]

Thomas is here speaking about two types of validation that take different inputs and take place at two different levels. It appears that one primary purpose with the content at Level I is to enable the reader to validate the proofs using a type of higher-level validation that is different from line-by-line validation.[6] Note that validating or verifying a proof using higher-level validation implies understanding what is going on in the proof. For Thomas, this is closely tied to the "visualization" of the central concepts in the proof.

Thomas gave a description of line-by-line validation when describing how he himself validated and revised a certain proof from Adam's initial draft. He explained that,

A high level of detail was necessary for me to understand Adam's proof. At first, I had no overview of the proof's overall argument and had to therefore focus on every single detail and on verifying, somewhat blindly, each logical step. Once I

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[5] He added that,

This approach works well for me and is quite effective in weeding out errors. However, I could imagine that other mathematicians might have other approaches, some of which are much more detail-oriented and less narrative or pattern based.

[6] We take Thomas' response to imply that a proof typically cannot be validated using only higher-level proof validation. Even when higher-level proof validation indicates that the proof is correct, "some additional checking of the more oblique or technical parts" is typically called for, although the patterns of the proof, including the patterns of these parts, have been validated. It is not clear how "the more oblique or technical parts" of the proof are identified.

> *had visualized a clear picture of the proof, I could then make a few small modifications to make the flow of logic more streamlined.*

Andersen (2018) reports on interviews with mathematicians about their refereeing practices. The interviews suggest that a referee validates some of the subproofs in a submitted paper by holding the subproofs, considered in broad outline, up against the landscape of mathematical knowledge. Other subproofs are validated line by line. Furthermore, the interviews suggest that a referee, when she checks a proof for correctness, also checks whether other experts would be able to check the proof for correctness in the same type of way. So, like Thomas does as the author of a paper, the referee apparently aims for the proofs in the paper under review to be prepared for both higher-level validation and line-by-line validation by the intended audience. In this sense, the two studies complement each other nicely. Andersen (2018) does not go into detail with *how* proofs are prepared for higher-level validation, nor does she at all examine how proofs are prepared for line-by-line validation, which is what we will do in the next section.

5. Removing Narrative Voice to Make Proof Structure Clearer

We observed that the proofs in the last version of Adam and Thomas' paper have no narrative voice that explains why the authors proceed the way they do or contextualizes the results. For example, the first draft included a sentence beginning with "If it can be shown that…," which explained why showing that *q* is a step in the direction of showing that *p*, which is the result, they want. This explanation preceded the proof of *q*, but the sentence did not appear in the last version of the paper. This section focuses on the purposes of this lack of explanatory or contextualizing voices. The immediate purpose seems to be to make the proofs more formal and make the logical structure and details of each proof stand out more clearly. This can be illustrated with Thomas' answer to the following question,

> *This question is about the change you made to the part of the proof of theorem 0.10/2.3*[7] *that comes after the proof by induction. We call this part of the proof subproof 2. You have removed a narrative voice from subproof 2 that presents the proof as originally thought. By this we just mean that in [the first version of the paper] the proof by contradiction is given after you have presented your reason*

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[7] 0.10 and 2.3 indicate the number of the theorem in the first and the last version of the paper, respectively.

for needing the result proven by contradiction, whereas the proof by contradiction comes first in [the last version of the paper].

By removing this narrative voice, you seem to aim to better or faster convince the reader that the result proven by subproof 2 is correct. If this is accurate, please describe why you took this action to further this aim. If not, please describe what your aim was with moving the proof by contradiction down.

Thomas responded,

I removed some lines, namely those starting with "If it can be shown …" Providing an outline of some line of argument that is to follow is often useful to an audience but the level of detail in this instance was more suitable for a lecture presentation, or perhaps even a thesis. Here, I felt, it gave better clarity for the reader to have a streamlined proof with a clear and flowing sequence of arguments, rather than back and forth meta-commentary.

There is also some convention at play here: in lectures and seminars, it is usual to hear the use of 'We,' partly by tradition but also since it invites the audience to feel included to participate, thus making a more engaging lecture but also to form an alliance that might help weaknesses of presentation to be forgiven or even overlooked.

This convention sometimes spills over into written mathematics. […] [But] to me, though this might not be true of other mathematicians, the avoidance of 'We' and 'I' in proofs makes proofs easier to read and visualize, since the simplicity thus enhanced makes the logic clearer to see.

Consequently, an immediate purpose of removing explanatory or contextualizing voices from the proofs seems to be to make the proofs more formal and impersonal, and to make the logical structure and details of each proof stand out more clearly (concrete examples of what is meant by 'structure' here are given below). In a couple of places, Thomas uses the verb 'streamline' when describing this process, reporting that his purpose was "to make the flow of logic more streamlined" (quoted above) or "to make the logical arguments […] more streamlined." Similarly, he writes that he wanted "to introduce […] a clear proof structure in which the logical arguments were clear and appeared in orderly sequence."

Thomas' answer also implies that the proofs are not only made more formal by leaving only logical structure and details, but also by limiting the 'We' of the authors or the authors and the reader. The pronoun 'we' appears in just over half of the proofs in the first version of Adam and Thomas' paper and in just over a third of the proofs in the last version. Hence, it is not only the authors' *explanatory* voice that has been removed, but to some extent their voice in general. In this sense, the authors are more present at Level I, and thus there is greater distance between the authors and the text at Level II. When asked about the persistence of 'we' in the Level II text, Thomas explained that *within* the proof context, he preferred to use 'we' to introduce meta-commentary – such as motivation and overview – in *long* proofs. Thus, the pronoun could be used essentially to provide Level I 'anchors' within long Level II passages of the text.

To give an example of how a proof is revised to better bring out the structure, we quote Thomas' response to another question about the same proof. Thomas had not only deleted parts of the proof, but also added details to the proof. On this, he wrote,

> *The conclusion by induction on the size of the family in question […] deserved, I felt, its own separate sentence, to make this final argument clearer to the reader and to also indicate the proof structure more clearly, here that this was the end of this part of the proof. I also added the detail "by assumption" which I felt should be expressed explicitly for better clarity, even though most readers would presumably be quickly able to infer that detail implicitly.*

Hence, in this example, the proof structure was made clearer by making it more transparent where a subproof by induction ended and by making explicit the proof scheme by indicating where the induction hypothesis was used as an assumption in the proof.

It seems that logical detail was also sometimes removed from a proof in order to bring out the logical structure or to "streamline" the proof. This can be seen in an example that we pointed out in one of the interview questions. One part of a proof looked as shown in Figure 3 and ended up looking as shown in Figure 4.

> As $\mathscr{F}(\bar{n})$ is just a subset of $\mathscr{F}$ it also a $(d, 2k-(d-3))$-conditionally intersecting k-hypergraph. As $\mathscr{F}$ is stable in particular for any shift S_{ij} with $j \neq n$ and set $F \in \mathscr{F}(\bar{n})$ not containing n either $i \in F$, $j \notin F$ or the set $(F-\{j\}) \cup \{i\} \in \mathscr{F}$ and as this does not contain n it is in $\mathscr{F}(\bar{n})$. Thus $\mathscr{F}(\bar{n})$ is stable and by induction $|\mathscr{F}(\bar{n})| \leq \binom{n-2}{k-1}$.

Figure 3.

> Since $\mathcal{F}(\bar{n}) \subseteq \mathcal{F}$ and since $\mathcal{F}$ is stable and satisfies Condition (3), the family $\mathcal{F}(\bar{n})$ must also be stable and satisfy Condition (3). Thus by assumption, $|\mathcal{F}(\bar{n})| \leq \binom{n-2}{k-1}$.

Figure 4.

If the *immediate* purpose of removing explanatory or contextualizing voices from the proofs, and of the authors to some extent removing themselves from the proofs, is to make the proofs more formal and make their logical structure and details stand out more clearly, what is the *higher* purpose of this? When a reader validates a proof using higher-level validation, she (by definition) does not need all the details of the proof, but the previous section suggests that the structure of the proof is important. If this is true, making the *structure* of the proof stand out more clearly makes higher-level validation easier for the reader. Yet, it cannot explain the authors' partial removal of themselves from the proof and the emphasis on the logical details.

Instead, this presumably has to do with line-by-line validation. When a reader tries to validate a proof using higher-level validation, she does so under the guidance of the authors in the sense that they have taken her through and familiarized her with the relevant concepts and approaches in ways they deem appropriate. If her attempt to validate the proof in this way leaves her skeptical of the proof and thus the guidance of the authors, she will want to check it by herself, without their guidance. For example, she may suspect that the intuitions about some concept provided at Level I are unreliable because there is a danger that the concept will behave counterintuitively in the context in which it appears in the proof. This may explain why Adam and Thomas to some extent remove their own voice from the proof and let the logical structure and details speak for themselves. The logical structure and details are what the reader needs to validate the proof using higher-level validation.

6. Discussion

We have seen how Adam and Thomas' paper has gone through a process that made the paper address the intended audience in a complex way. Hence, the completed paper addresses the audience in one way at Level I – i.e. in the parts of the paper outside the Definitions, Propositions, and Proofs – and in another way in the Proofs at Level II. Adam and Thomas have made *additions* to Level I of their paper with the purpose of taking the reader by the hand; showing her that the results are interesting and leading her through the proofs in outline. By contrast, one purpose with the *deletions* at Level II is apparently to leave the reader to validate the proofs by herself, line by line.

On this picture, information external to the designated proof parts of the paper plays a central role in the validation of the proofs by enabling a certain higher-level validation. The implications of this are captured and emphasized by the following account of the structure of Adam and Thomas' paper.

We may say that Adam and Thomas' paper has been revised to contain two mathematical arguments for the same results. The two arguments take place at two different levels: one takes place at Level I and also contains the structure and results of the proofs at Level II, the other takes place at Level II (see Figure 5). The two arguments are parallel in the sense that the main structure of the proofs and the results and main subresults in the proofs are contained in both arguments, yet importantly neither level is reducible to the other one: The Level I argument does provide "sugar" to attract interest, but it also provides indispensable context and anchoring for the Level II argument.

One may think of the two arguments as two parallel interrupted lines signifying the reading process, where the extremes of each line represents the beginning and the end of each argument. The lines have points drawn on them, representing steps in the arguments. Points at Level I may correspond to a series of points at Level II which can be seen as expansions of the argument into higher levels of granularity. By moving between levels, the reader may easily switch between modes of reasoning when in need of checking some result by checking its proof line by line.

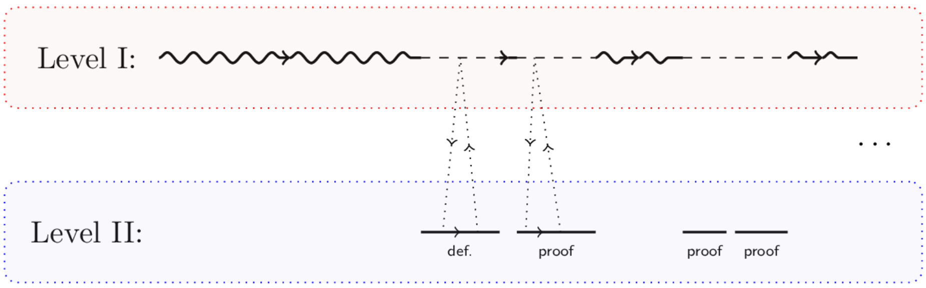

Figure 5. Schematic illustration of the text of a mathematical paper separated into Level I and Level II parts. The figure is intended to show the reading process where certain parts of the Level II text may be skipped because a comprehension of the proof has already been achieved, mainly from the Level I text.[8]

Thus, the two parallel arguments as written have different levels of granularity. In particular, the proofs present themselves to the reader at different levels of granularity depending on whether the reader is reading

[8] The exact content of Level I is not given by the presentation of the proof, and Level I will not look the same to every reader. The amount of detail at Level I depends on the reader. Hence, Figure 5 represents Level I and Level II relative to a given reader and allows for some variation. If we wanted to represent Level I relative to the general audience of the proof, we would have needed more lines.

the Level I argument or the Level II argument. This is not only true of the two arguments as they appear on paper, but also as they appear in the minds of the mathematicians validating them. Mathematician and philosopher Yehuda Rav wrote that, when one reads a proof, it "often happens – as everyone knows too well – that one arrives at an impasse, not seeing why a certain claim *B* is to follow from claim *A*, as its author affirms." In such situations, "one picks up paper and pencil and tries to fill in the gaps" by reflecting "on the background theory [and] the meaning of the terms," and by "using one's general knowledge of the topic" (1999, p. 14; cf. Rav 2007, pp. 216–217). When mathematicians are described as using intuitions and background knowledge to fill in the gaps, we would say that they are validating Level II arguments. However, mathematicians also use intuitions and background knowledge when validating Level I arguments, but *without* filling in gaps since they are comparing the flow of the argument to what they know and understand in advance. In Adam and Thomas' paper, these intuitions and background knowledge are partially provided to them by the Level I argument itself.

In light of this partial account of the structure of Adam and Thomas' paper, it is worth revisiting the question of how the relationship between the authors and the reader are different at Level I and Level II. In the Level I argument, the authors try to provide the reader with and elicit in the reader certain intuitions that support the conclusions they want to get to. In this sense they take the reader by the hand and lead her through the Level I argument. By contrast, they leave the reader to validate the Level II argument on her own, using intuitions that have not been provided by the authors in the Level II argument to fill in the gaps. It should be stressed that this does not imply that the reader depends on the authors for the correctness of the Level I argument. The intended reader is an expert who is capable of evaluating independently the argument and the tools provided to her by the authors. If she becomes suspicious, she may validate the suspicious part of the Level I argument by validating the corresponding part of the Level II argument.[9]

7. Conclusions

Through interviews with the talented PhD student Adam and his supervisor, the experienced mathematician Thomas, about their revisions,

[9] Consequently, when a reader does not check a proof line-by-line this does not imply that she is just trusting the author to have done so. She may have validated parts of the proof line-by-line and validated the other parts by independently validating parts of the Level I argument.

we have gained insight into an important part of mathematical practice that is often kept private. Adam had authored a first draft on his own, with good expositional skills but without the benefit of experience in writing a research paper for a mathematical audience and, therefore, without the knowledge of what such audience might expect and prefer. For example, when asked about why he presented a proof a certain way in the initial draft, Adam explained that he was motivated by "understanding" the proof *himself*, and that "there was no motivation to 'convince' [the reader] my proof was correct." Our interviews and analyses have made it clear how this lack of attention for the reader manifested itself in the text, and contrasting Adam's first draft with the final paper has made it possible to see some of the central elements that are at play when you write for a mathematical audience: In the first draft of the paper, Adam primarily made a Level II argument, although he included some Level I elements in it. The fact that Adam primarily made a Level II argument was presumably due to Adam's training as a mathematics student, in which he likely checked proofs line-by-line (Inglis & Alcock 2012) and learned how to construct proofs from this practice. His Level II argument was clarified to the extent that it contained an explanatory voice which, according to Thomas, is a kind of voice that is rarely suitable at that Level II in mathematical research papers. Under Thomas' guidance, Adam and Thomas revised the paper to contain a Level I argument, setting up a structure of dual arguments that allow for reading and validation at different levels. Thus, Level I is more amenable to epistemic validation and communication of understanding, whereas Level II is tailored for more formal validation. This way, the paper was made to correspond to the more tacit requirements raised in mathematical practice that may contrast the false ideal of logical proof-steps as the exclusive content of mathematical meaning. And Thomas was quite explicit and reflected about this motivation for revising the paper; in fact, giving guidance to Adam through the process.

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