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Andersen, Line Edslev

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19 Reading Mathematical Proofs as Narratives

Line Edslev Andersen

Abstract
Mathematical proofs and narratives may seem to be opposites. Indeed, deductive arguments have been highlighted as clear examples of non-narrative sequences by narrative theorists. I claim that there are important similarities between mathematical proofs and narrative texts. Narrative texts are *read* in a quite distinct way, and I argue that mathematical proofs are often read like narrative texts by research mathematicians. In this way, narratives play an important role in mathematical knowledge-making. My argument draws on recent empirical data on how mathematicians read proofs. Furthermore, my examination of mathematical proofs and narratives provides an account of what it means for research mathematicians to *understand* mathematical proofs.

19.1 Introduction

Mathematicians sometimes emphasize the major role of *inductive* reasoning in mathematics (see, for example, Borwein and Bailey 2003). Results in mathematics are usually tested in reliable ways before a *deductive* mathematical proof of the results is produced. For example, the famous Riemann hypothesis has been verified in billions of instances. But results in mathematics are established by deductive mathematical proofs, and the Riemann hypothesis will remain just that, a hypothesis, until a deductive mathematical proof has been produced. Historically, deductive mathematical proof has become the ultimate method of justification in mathematics. For this reason, proofs play a very central role in mathematical research practice.

Mathematical proofs and narratives may seem to be opposites. Indeed, deductive arguments have been highlighted as clear examples of non-narrative sequences by narrative theorists (see, for example, Bruner 1987: 11–14; Herman 2009: 157). By contrast, scholars interested in the nature of mathematical proof have occasionally conceived of mathematical proofs as narratives (Doxiadis 2012; Robinson 1991: 269; Thomas 2007). Their accounts mainly focus on the similarities between mathematical proofs *as written* and narrative texts.

I similarly compare mathematical proofs with narratives but take a different approach.[1] Narrative texts are read in a quite distinct way, and I argue that mathematical proofs are often *read* like narrative texts by research mathematicians.[2] Hence, my account of mathematical proofs and narratives mainly focuses on the relationship between mathematical proofs and their readers rather than their writers.[3] The argument constitutes an independent argument for conceiving of mathematical proofs as comparable with narratives, and sheds new light on the reading of mathematical proofs. My argument draws on recent empirical data on how mathematicians read proofs.

Furthermore, my account of mathematical proofs and narratives provides an account of what it means for research mathematicians to *understand* mathematical proofs. This account of proof understanding appears to have implications for how we should conceive of proof validation and for how proofs should be taught.

19.2 Reading Proof as Narrative

Before I say more about how proofs are read, I should say a few words about how proofs are *made* and *presented*. In recent years, philosophers have developed accounts of how we should conceive of a mathematical proof as a sequence of inferential *actions* performed by an agent on various objects, such as propositions, diagrams and mental images (De Toffoli and Giardino 2015; Larvor 2012; Netz 1999: 51–56; Tanswell 2017a; 2017b: 144–153). For a vivid image of a proof involving inferential actions performed on diagrams, one may watch one of the many videos on YouTube of different diagrammatic proofs of the Pythagorean Theorem (or see Tanswell 2017a: 223–225).

If a proof is a sequence of actions, we may conceive of a proof *as written*, as found, for example, in a research article or a textbook, as a telling of a sequence of actions. Hence, a written proof is a telling of how something happened. A written proof is a telling of a sequence of actions performed on mathematical objects by an agent with the aim of proving a given proposition (in line with Hamami and Morris 2020). This implies that a proof as written is a *narrative* in at least a minimal sense. A narrative is often minimally characterized as a telling of a sequence of particular events or actions involving humans or humanlike characters with particular goals (see, for example, Sanford and Emmott 2012: 1–5; Toolan 2001: 4–8). To be more precise, a written proof is

[1] On the role of narrative in scientific reasoning more broadly, see Morgan (Chapter 1).

[2] In the introductory chapters in this volume, Morgan (Chapter 1) and Hajek (Chapter 2) distinguish between two broad senses of narrative in relation to science: narrative representation and narrative reasoning. My argument is about narrative reasoning.

[3] For a broader account of the relationship between narration in science and the reader, see Hajek (Chapter 2).

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a telling by the author of a sequence of actions on mathematical objects performed by an agent, usually by the author of the proof herself, with the aim of establishing a particular theorem (this is in line with Doxiadis 2012: 330–331, on Euclidean proofs).

I ask readers to have something like this picture of proofs in mind as they read on. But rather than focusing on the narrative features of written proofs, the aim of this chapter is to argue that written proofs can be and often are *read* like narratives by research mathematicians. A narrative is not only characterized by features of presentation. Narratives are also characterized by how they are read or heard. In fact, narratives have a quite distinct relationship with their readers or auditors. This relationship is sometimes described by narrative theorists as having two key features (see, for example, Bruner 1991: 11–13; Herman 1997).

The first key feature of the relationship between narratives and their readers or auditors is that narratives are interpreted by their readers or auditors against the background of patterns of belief and expectation with respect to how events or actions of the kinds represented in the narrative usually take place. The background patterns of belief and expectation are called *scripts* by theorists like David Herman and they stem from the prior experience of the readers.[4] Hence, scripts describe *standard* sequences of events or actions against which a *particular* narrative is read. The narrative cues readers to activate the scripts.[5]

Consider, for example, the following narrative or part of a narrative: 'John went to Bill's birthday party. He watched Bill open his presents. John ate the cake and left' (adapted from Schank and Abelson 1977: 39). We read this particular narrative against our birthday party script which describes standard sequences of events and actions that usually take place at birthday parties in our experience. About similar examples of narrative sequences, Herman (1997: 1051) writes: 'I can make an astonishing number of inferences about the situations and participants – fill in the blanks of the stories, so to speak – because the sequences unfold against the backdrop of the familiar birthday-party script'. For example, when we read the narrative about John against our birthday party script, we can fill in sequences of actions such as John congratulating Bill upon arriving at his house, John giving Bill a present, and Bill tearing the wrapping paper before opening the box.

This conception of how narratives are read or heard implies that readers *reconstruct* a detailed story from a narrative text with less detail. It is important to note that readers reconstruct the actions behind the narrative text with the way events or actions usually occur based on the reader's prior experience. This is to say that scripts vary across readers whose prior experience is substantially

[4] I am grateful to Kim M. Hajek for alerting me to the research on scripts and its potential for work on mathematical proofs.
[5] On the value of the notion of script to the Narrative Science Project, see Hajek (Chapter 2).

different. Different readers may also make different reconstructions when they have similar scripts available if they make different assumptions about the context of a particular narrative. Thus a reader who assumes John and Bill to be children would fill in a sequence where Bill's mother lit candles on the cake, Bill blew out the candles, then his mother cut the cake and handed a slice to John. Another reader, assuming Bill to be an older man, might imagine there to be no candles, and Bill cutting the cake himself. An author who wanted to specify who cut the cake would need to write it out explicitly as part of the narrative sequence, not just rely on the birthday party script to do that job.

The process by which readers or auditors activate their prior experience captured in scripts when reading or hearing a narrative is sometimes described as the process by which they come to *understand* the narrative. I will return to the topic of understanding towards the end of this chapter.

The second key feature of the relationship between narratives and their readers or auditors is that readers or auditors are usually *surprised* by parts of the narratives.[6] The surprises are surprises against the backdrop of scripts. They are breaches of the scripts. In other words, the unusual or surprising aspects of narratives are unusual and surprising against the backdrop of the scripts. When the narratives convey something unexpected relative to existing scripts, the narratives can feed into new scripts. Hence, a narrative unfolds against the backdrop of scripts but also contributes to the creation of new scripts. Existing scripts are exploited to generate new scripts.

For example, we may add to the birthday party narrative featuring John and Bill that Bill's cat tried to lick John's hand, attracted to the cheese that had oozed out as John bit into the cake. There is a breach of the birthday party script here, as we expect a birthday cake to be made of flour, sugar, eggs and so on, not cheese. The breach of the birthday party script may lead us to reconsider the new trend of using cheese rounds as cakes, and when we on several occasions have read or heard narratives where scripts are breached by involving cheese rounds as cakes, we may be led to new scripts about usual sequences of actions in which a cake made of cheese rounds is served.

The two key features of the relationship between narratives and their readers or auditors are closely related. Herman (1997) describes the relationship thus: 'Stories stand in a certain relation to what their readers and auditors know, focussing attention on the unusual and remarkable against a backdrop made up of patterns of belief and expectation. Telling narratives is a certain way of reconciling emergent with prior knowledge' (Herman 1997: 1048). Focusing

[6] Netz has written on narrative and narrative surprises in mathematics. He writes about the narrative structure of Greek mathematical treatises with a particular focus on narrative surprises (Netz 2009: 80–91). See also Hurwitz (Chapter 17) on narrative surprises in medical anecdotes.

on social science research, Morgan (2017) similarly emphasizes the importance of narrative in framing and resolving puzzles.

I claim that research mathematicians often read proofs as narratives. Hence, I claim that proofs are often read as narratives in addition to having the presentational features of narratives by being a telling of a sequence of actions. More specifically, proofs are often interpreted by their readers against the background of scripts. The scripts are about how different kinds of results or sub-results are usually proved, about standard sequences of proving actions; and the scripts are based on the experience with proofs of the research mathematicians.[7] Furthermore, the readers focus attention on the breaches of the scripts, on that which appears unusual or surprising against the background of the scripts.

For example, consider the following telling of a sequence of actions aimed at proving the formula $1 + 3 + 5 + \ldots + (2n - 1) = n^2$, which holds for any natural number n. I begin by showing that the formula holds for $n = 1$. $((2 \times 1) - 1)$ equals 1, which, in turn, equals 1^2 and thus the formula holds for $n = 1$. I now show that *if* the formula holds for some value $n = n_0$, *then* it also holds for $n_0 + 1$. Hence, I make the assumption that the formula holds for $n = n_0$, that is, I make the assumption that $1 + 3 + 5 + \ldots + (2n_0 - 1) = n_0^2$. Given this assumption, I have to show that the formula holds for $n = n_0 + 1$, that is, I have to show that $1 + 3 + 5 + \ldots + (2n_0 - 1) + (2(n_0 + 1) - 1) = (n_0 + 1)^2$. Using the assumption, I get that $1 + 3 + 5 + \ldots + (2n_0 - 1) + (2(n_0 + 1) - 1) = n_0^2 + (2(n_0 + 1) - 1)$. And, simplifying, I get $n_0^2 + (2(n_0 + 1) - 1) = n_0^2 + 2n_0 + 2 - 1 = n_0^2 + 2n_0 + 1 = (n_0 + 1)^2$. In short, $1 + 3 + 5 + \ldots + (2n_0 - 1) + (2(n_0 + 1) - 1) = (n_0 + 1)^2$, which is what I wanted.

If we have some experience reading proofs by mathematical induction we may interpret the telling of the sequence of actions against the background of what we may call the proof by mathematical induction script, that is, against the standard sequence of actions performed in proofs by mathematical induction. We will then see that the particular sequence of actions taken in the example follows the script. In fact, in this example, we need to read the telling of the sequence of actions against the background of this script or pattern of actions in order to see that the formula has been proved. An analogous point can be made if we consider the following narrative: ‘John went over to Bill’s house. He watched Bill open his presents. John ate the cake and left’ (adapted from Schank and Abelson 1977: 39). Readers will see that the description fits the birthday party script and can fill in the important ‘detail’ that Bill is having a birthday party. In short, relying on scripts, readers recognize that a standard proof by mathematical induction is taking place without being told so explicitly

[7] Hopkins (Chapter 4) similarly emphasizes the role of the training and experience of geologists in how they read narrative texts in geology.

and, similarly, readers recognize that a birthday party is taking place without being told so explicitly.

In the particular case of the example of the proof by mathematical induction, there are no surprises when we read the telling of the sequence of actions against the background of the proof by mathematical induction script. This is to say that the sequence of actions in the example proceeds entirely as we expect given our experience with how proofs by mathematical induction are usually carried out. But we would probably have been surprised or paid special attention if an unusual idea were used to show that the formula holds for $n = n_0 + 1$ or if the formula were of a kind where we would not expect the formula to be provable by mathematical induction.

It is worth emphasizing that my claim that proofs are often read as narratives is a claim about how research mathematicians read proofs. My discussion of the proof by mathematical induction thus provides a simplified illustration of how research mathematicians read proofs, since I speak about how we, who are not research mathematicians, would read the proof by mathematical induction. Most of us possess only few and simple scripts about standard sequences of actions in mathematical proofs.[8]

19.3 Evidence

In this section I draw on recent interview studies with research mathematicians about how they read proofs. The interview data is consistent with the claim that mathematicians can and often do rely on scripts when they read proofs, and that they focus attention on the breaches of the scripts, on that which appears unusual or surprising in the proofs against the background of scripts.

It is important to note that how mathematicians read proofs may well have changed substantially over time. And even if we assume that mathematicians now tend to read proofs as narratives and always have tended to read proofs as narratives, the scripts they have used will have changed over time. For

[8] My account of how proofs are read focuses on *actions* on the part of the authors and the readers of proofs. Previous accounts of mathematical proofs that focus on the actions on the part of the readers of proofs have conceived of a proof as a *recipe* of sorts for how to prove a proposition (Larvor 2012: 725–726; Sundholm 2012; and Tanswell 2017b). They claim that a proof as written is a recipe for how to execute an actual proof. Reading a proof is like reading a recipe and the readers are supposed to follow the recipe and perform steps of the actual proof as they read the proof recipe. My account of how proofs are read as narratives and the account of proofs as recipes are not necessarily inconsistent. It is possible that the two accounts capture different aspects of proof reading. In any case, the two accounts emphasize different kinds of actions on the part of the readers. When we conceive of a proof as a recipe, the action on the part of the readers of performing steps in the proof is emphasized. By contrast, when we conceive of reading a proof as reading a narrative, then the action on the part of the readers of connecting steps in the proof to scripts, of recognizing the steps performed by the author of the proof as instances of scripts, is emphasized.

example, the fact that mathematicians have used different criteria for mathematical proof across cultural and historical contexts means that they will have used different scripts. I here rely on research in the history and philosophy of mathematics which has uncovered how mathematical proofs as they occur in mathematical practice are context-sensitive. For example, the level of rigour required for mathematical proof has varied across time and discipline.

19.3.1 Reliance on Scripts

Various recent interview studies with research mathematicians suggest that mathematicians read proofs against the background of what they know from their experience with proofs. For example, these studies suggest that mathematicians have beliefs and expectations about which methods and techniques work in which situations and on which sorts of mathematical objects, and that mathematicians rely on these beliefs and expectations as they read proofs. The mathematicians seem to see recognizable patterns of action in the sense of standard sequences of proving actions in the proofs. In this sense, proofs seem to unfold against the backdrop of scripts about standard sequences of proving actions. In sum, mathematicians, as they read proofs, seem to rely on scripts about which methods and techniques work in which situations and on which sorts of mathematical objects.

For example, based on their interviews with mathematicians, Weber and Mejía-Ramos (2011: 340) suggest that mathematicians, when they read proofs, 'might encapsulate strings of derivations into a short collection of methods and determine whether these methods would allow one to deduce the claim that was proven'. Whether the methods will work to prove a result is something they judge based on their experience. Weber and Mejía-Ramos note (2011: 340) that Konior (1993) provides further data to support their claim. Konior reports on the analyses of several hundred proofs. Konior found that a written proof often contains cues that indicate to the reader how to separate the proof into parts and what methods were being used in each part. For example, a part of a proof may begin with: 'We have to define a one-to-one mapping *g* of *X* onto *Y*' (Konior 1993: 255). In this way, a proof seems to cue readers to activate their scripts about which methodological moves work when.

Andersen (2020) has interviewed mathematicians about their proof reading practices when they act as referees for mathematics journals. Based on the interviews, she similarly suggests that mathematicians read proofs against their experience concerning which approaches work to prove different kinds of results. Mathematicians appear to have reliable intuitions based on experience about which type of approach can typically be used to prove a sub-result of this or that type. The beliefs and expectations the interviewees have about proving actions seem to correspond to what we here call scripts.

A study by Andersen, Johansen and Sørensen (2019) indicates that the scripts may in part be provided by the main text preceding a proof in an article. The study reports on interviews with a supervisor and his PhD student. In line with the interviews referenced above, the supervisor described how he studies the ‘general pattern’ or flow of a proof and sees if it is recognizable or instead raises ‘red flags’, which can be interpreted to mean that he studies whether the proof follows the scripts or there are points at which a script is breached. He pays special attention ‘if some of this pattern recognition raises a red flag or indeed gives any hint of unease or alienation’ (Andersen, Johansen and Sørensen 2019: 11). His expectations with respect to the flow of the proof are sometimes informed by the main text of the article presenting the proof. He emphasized how an article may provide examples of how a mathematical object behaves in different situations before presenting proofs establishing results about the object in question. The examples may then influence the expectations of the supervisor with respect to how the results can be proved.

The interview data is consistent with my claim that mathematicians can and often do rely on scripts based on their experience as mathematicians when they read proofs. The interview data does not shed light on the question of what concrete scripts that play a role in mathematical practice look like exactly. This is a question for future research. Note that we would expect that the parts of proofs that follow the scripts are *not* the parts readers focus attention on, exactly because there are no breaches of the scripts. This is supported by Andersen, Johansen and Sørensen (2019) and Andersen (2020), whose interviews suggest that mathematicians do not thoroughly read the parts of a proof that unfold the way they would expect.

19.3.2 Breaches of Scripts

Sometimes something unusual happens in a proof. In a number of interview studies, mathematicians describe how they pause and pay close attention when they read a proof and encounter something ‘surprising’ (Andersen 2020: 238) or something ‘strange’ or ‘odd’ (Weber 2008: 448). Or how they pay close attention when a part of a proof is ‘suspicious’ (Müller-Hill 2011: 307–308, 327–328) or raises a ‘red flag’ (Andersen, Johansen and Sørensen 2019: 11). My argument above offers an interpretation of the parts of proofs the mathematicians describe here. I argued that the *usual* moves in proofs can be interpreted as the moves that follow the scripts about standard sequences of proving actions. The *unusual* moves that mathematicians describe that make them pause and pay close attention when they are reading proofs should then be interpreted as the moves that do not follow the scripts about standard sequences of proving actions. The unusual moves are breaches of the scripts.

As described in section 19.2, ‘Reading Proof as Narrative’, above, when narratives convey something unexpected relative to existing scripts, the narratives feed into new scripts. In the case of mathematical proofs, we may ask how unusual moves contribute to the creation of new scripts. Moves that may be unusual at one point in time may become standard moves at a later point in time because they have been shown to work in various proofs.[9] Before new moves can turn into standard moves, the new moves must be carefully checked in the proofs that use them. This probably involves careful attention to detail and filling intentional gaps with extra details.[10] It has previously been claimed that readers of mathematical proofs commonly fill intentional gaps in the proof and thus engage in a kind of *reconstructio*n of the proof (see, for example, Fallis 2003; Netz 2009: 71–80; Rav 1999).[11] It is worth adding that how the readers perceive the *narrator* or the author of the proof may affect how thoroughly they check unusual moves. Readers may be more thorough if the author is a PhD student than if the author is an experienced mathematician (Andersen 2017: 184–187; Inglis and Mejía-Ramos 2009; Mejía-Ramos and Weber 2014: 165–168).

When proofs are read as narratives, reading proofs really involves two kinds of *reconstruction* of the proofs on the part of the readers, one of which is the kind of reconstruction of proofs that has previously attracted attention from philosophers. Readers of proofs engage in the kind of reconstruction that has been discussed previously when they *fail to see* what is going on, when they cannot follow a step in a proof, that is, when they cannot see why B follows from A as the author claims. The readers will then insert extra steps between A and B. As just mentioned, this kind of reconstruction probably plays a role where breaches of scripts occur and in establishing *new* scripts. But readers also engage in a kind of reconstruction when they *recognize* what is going on, when they recognize a move in a proof as an instance of a script for a standard way of proving this sort of result. The details they insert in the proof are provided by

[9] This process is similar to the process described in Morgan (2005: 324) of how a ‘surprising behaviour pattern’ observed in an experiment in economics may turn into a ‘genuine behaviour pattern’ over time, ‘after many experimental replications with many subjects and with slight variations in the experimental design’.

[10] Jajdelska (Chapter 18) demonstrates a different way in which readers of research articles are led to accept unusual ideas presented in the articles: through narrative performativity. Meunier (Chapter 12) demonstrates how readers of research articles can be made familiar with new methods and epistemic objects by being guided through a narrative sequence by the authors.

[11] For example, Rav (1999) describes his experience with reading proofs. He writes that, when one reads a proof, it ‘often happens – as everyone knows too well – that one arrives at an impasse, not seeing why a certain claim q is to follow from claim p, as its author affirms’. Thus, ‘one picks up paper and pencil and tries to fill in the gaps’, both by reflecting ‘on the background theory [and] the meaning of the terms’ and by ‘using one’s general knowledge of the topic’ (Rav 1999: 14). Sørensen, Danielsen and Andersen (2019) provide an account of how this kind of reader engagement can be taught to students as an aspect of proof.

the *existing* script. Hence, in both kinds of reconstruction details are filled in by the readers but the details are different in kind and are inserted in different parts of the proof.

19.4 Understanding Proofs

The present account of how proofs are read provides a way of thinking about mathematical understanding, more specifically the understanding of proofs.[12] Among cognitive scientists, understanding is commonly characterized as 'a process by which people match what they see and hear to pre-stored groupings of actions that they have already experienced' (Schank and Abelson 1977: 67; quoted in Herman 1997: 1048). In particular, coming to understand a narrative is the process by which narratives are interpreted by their readers or auditors against the background of scripts about how events or actions of the kinds represented in the narrative usually take place. Hence, the process by which readers or auditors come to understand a narrative is the process by which they use scripts to reconstruct the narrative. Consider again the narrative: 'John went over to Bill's house. He watched Bill open his presents. John ate the cake and left' (adapted from Schank and Abelson 1977: 39). This narrative does not make much sense if we do not interpret the narrative against our knowledge of how birthday parties usually take place. We come to understand the narrative by reading the narrative against our birthday party script. Thus, we envision the guests arriving at Bill's house, the guests each giving Bill a present, and Bill tearing the wrapping paper.

When the narratives convey something unexpected relative to existing scripts, the readers can fail to understand. For example, in the case of the narrative where Bill's cat tried to lick John's hand, the readers may fail to understand how John ended up with cheese on his hand when he ate cake, since, according to their birthday party script, a birthday cake is made of flour, sugar and eggs, not cheese. But, when the narratives convey something unexpected relative to existing scripts, the narratives can also contribute to the creation of new scripts and thus new 'models for understanding' (Herman 1997: 1056). Hence, when the readers see that Bill's birthday cake is made of cheese rounds, and on a number of other occasions have read narratives where scripts are breached by involving cheese rounds as cakes, they may be led to new scripts about usual sequences of actions in which a cake made of cheese rounds is served.

[12] Avigad (2008) argues that we must consider mathematical understanding of different things, such as theories, theorems and proofs, separately. Sandborg (1997: 140–141) discusses the difference between mathematical understanding of theorems and proofs.

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If proofs are read as narratives, as I suggest, the process by which mathematicians come to understand proofs is the process by which they use scripts to reconstruct the proofs. Thus, the process by which mathematicians come to understand a proof involves recognizing the moves in the proof as instances of scripts about standard sequences of proving actions.[13] Hence, the process by which mathematicians come to understand the proof of the formula $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ I gave earlier involves recognizing my moves in the proof as instances of the proof by mathematical induction script.

Consider the following useful analogy suggested by Norton Wise between coming to understand a proof and coming to understand how to frame a new roof. Coming to understand how to frame a new roof requires experience with patterns of roof framing in many particular instances. Considering only how each part of the new roof framing is placed is not enough for understanding how the stability of the whole emerges. Similarly, going through the steps of a given proof is not enough for understanding the proof but requires experience with patterns of proving in many particular instances. In other words, understanding the proof requires scripts about standard sequences of proving actions.

The present account of proof understanding can explain why mathematicians emphasize that one may have verified every logical step of a proof and still not have understood the proof. Poincaré makes this point. A mathematician may have 'examined [the elementary] operations one after the other and ascertained that each is correct' and still not have 'grasped the real meaning' of the proof (Poincaré 1958: 217–218). Feferman similarly notes that, 'It is possible to go through the steps of a given proof and not understand the proof itself', and adds that understanding the proof is 'a special kind of insight into how and why the proof works' (Feferman 2012: 372; quoted in Folina 2018: 136).

While verification is not a form of understanding, the opposite may be true. Understanding may be a form of verification. Research on proof reading tends to focus on the validation of proofs rather than the understanding of proofs. But the present account of mathematical understanding seems to suggest that there is a strong connection between the understanding and validation of proofs (in line with Dutilh Novaes 2018; and Mejía-Ramos and Weber 2014). The present account of understanding indicates that mathematicians *understand* proofs through action pattern recognition, which is the same kind of action pattern recognition that previous studies, based on interviews and a survey with mathematicians, suggest that mathematicians use to *validate* proofs (Andersen 2020; Andersen, Johansen and Sørensen 2019; Mejía-Ramos and Weber 2014; Weber and Mejía-Ramos 2011). Hence, proof understanding may be a form of proof validation.

[13] By contrast, Cellucci (2015) argues that understanding a proof consists in seeing how the different parts of the proof fit together.

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I end this section by briefly considering how the present account of mathematical understanding is relevant to mathematics education, to the teaching of proofs. As noted by Weber and Mejía-Ramos, ‘a goal of many research programs is to lead students to think and behave more like mathematicians with respect to proof’ (2011: 330). This raises the question of how students may read and come to understand proofs in a way that is similar to how mathematicians read and come to understand proofs. Needless to say, students will always have different scripts available to them than research mathematicians do. And presumably it is tempting for students to read proofs word by word without at all engaging in the kind of reconstruction of the proof scripts that mathematicians engage in. But the picture of proof reading presented in this chapter suggests that students cannot come to understand proofs this way. Not even mathematicians come to understand proofs by reading them only word by word rather than against the backdrop of scripts. Hence, if we want to teach students to read and come to understand proofs in a way that is similar to how mathematicians read and come to understand proofs, the teaching of scripts about standard sequences of proving actions is important. For example, it is valuable to teach students about the sorts of results that can be proved by mathematical induction and the commonalities between different proofs by mathematical induction.

19.5 Conclusion

Focusing on how mathematical proofs are *read*, I have argued that mathematical proofs can be and often are read like narratives by research mathematicians. Mathematicians read proofs as narratives when they read proofs against the backdrop of experience-based scripts about standard sequences of proving actions. They focus attention on the breaches of the scripts, on that which appears unusual or surprising against the backdrop of the scripts. The account I have defended of how proofs are read as narratives also provides an account of how to conceive of proof *understanding*, which is a topic that has received very little attention in the literature. On this account of proof understanding, a process by which mathematicians come to understand proofs is the process by which they relate proofs to scripts about standard sequences of proving actions.[14]

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