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PRESERVATION OF STOCHASTICITY AFTER PERTURBATIONS

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Working paper: Preservation of stochasticity after perturbations

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1 Introduction and problem setting

In [1], the SCEP-algorithm is introduced aiming to perturb a non-diagonalizable stochastic matrix $A$ in a diagonalizable stochastic matrix $\tilde{A}$ with preservation of spectral properties. Given a non-diagonalizable stochastic $3 \times 3$ matrix $A$ with real eigenvalues $1$ and $\lambda$, the SCEP-algorithm determines a matrix $E$ such that $\tilde{A}$ suffices the following conditions:

(a) $\tilde{A} = A + E$ is an additive perturbation of $A$.

(b) $\tilde{A}$ is a stochastic matrix.

(c) spec $\{\tilde{A}\} \subset \mathbb{R}$, $\mu \neq \nu$ and $\exists \delta > 0, \exists E \in \mathbb{R}^{3 \times 3}$ such that $|\mu - \lambda| < \delta \land |\nu - \lambda| < \delta$.

(d) The eigenspaces corresponding to the eigenvalue 1 of the matrices $A$ and $\tilde{A}$ coincide.

The conditions (a), (b) and (d) result in a perturbation matrix of the form:

$$E = \begin{pmatrix}
v_2(\epsilon_{22} + \epsilon_{23}) + v_3(\epsilon_{32} + \epsilon_{33}) & -v_2\epsilon_{22} - v_3\epsilon_{32} & -v_2\epsilon_{23} - v_3\epsilon_{33} \\
-v_1(\epsilon_{22} + \epsilon_{23}) & v_1\epsilon_{22} & v_1\epsilon_{23} \\
-v_1(\epsilon_{32} + \epsilon_{33}) & v_1\epsilon_{32} & v_1\epsilon_{33}
\end{pmatrix} \quad (1)$$

with four remaining perturbation variables $\epsilon_{22}, \epsilon_{23}, \epsilon_{32}$ and $\epsilon_{33}$ and $v = (v_1, v_2, v_3) \in V_L(1)$ a nonnegative left eigenvector corresponding to the eigenvalue 1. For condition (c), the directional derivative of the discriminant $\Delta$:

$$D_2\Delta(A) = (a_{12} - a_{13} - a_{22} + a_{33})(-v_2\epsilon_{22} - v_3\epsilon_{32} - v_2\epsilon_{23} - v_3\epsilon_{33} - v_1\epsilon_{22} + v_1\epsilon_{33})$$

$$+ 2(a_{13} - a_{23})(-v_2\epsilon_{22} - v_3\epsilon_{32} - v_1\epsilon_{23}) + 2(a_{12} - a_{23})(-v_2\epsilon_{23} - v_3\epsilon_{33} - v_1\epsilon_{23})$$

$$= KE_{22} + LE_{23} + ME_{32} + NE_{33} \quad (2)$$

must be positive.

In order to result in a stochastic matrix $\tilde{A}$, the perturbation matrix $E$ coming from the SCEP-algorithm should have positive matrix-elements corresponding to zero elements of the matrix $A$ and should have negative elements corresponding to the 1 elements of $A$. In this working paper, we verify whether every stochastic $3 \times 3$ matrix can be perturbed such that the result is again a stochastic matrix. The focus is on the 0 and 1 elements in a stochastic $3 \times 3$ matrix. To end up with a stochastic matrix $\tilde{A}$, the question is whether there exists a perturbation matrix $E$ with perturbations on the zero elements of $A$ that are positive, while the perturbations on the 1 elements of $A$ are negative.
We check this for all stochastic $3 \times 3$ matrices as follows. We partition the set of stochastic matrices, based on the number of zero entries in the stochastic matrices. A stochastic $3 \times 3$ matrix has 9 elements, therefore we consider the case of 9 zero elements up to the case of no zero elements in the stochastic $3 \times 3$ matrix $A$. In each case, we show that it is possible to perturb a stochastic $3 \times 3$ matrix such that all elements of the perturbed matrix $\tilde{A} = A + E$ are in the interval $[0, 1]$. This will be done in one of two ways: Either we present a concrete example of a perturbed matrix $\tilde{A}$ which suffices the conditions (a)-(d), or we prove there exists a perturbation matrix which satisfies the conditions (a)-(d).

## 2 Useful lemmas in reducing the number of cases

**Lemma 2.1.** If a non-diagonalizable stochastic $3 \times 3$ matrix $A$ contains a $2 \times 2$ submatrix without a zero element, then there exists a perturbation matrix $E$ which suffices the condition (a)-(d).

**Proof.** If a non-diagonalizable stochastic $3 \times 3$ matrix $A$ contains a $2 \times 2$ submatrix without a zero element, we will only consider the perturbation elements $\epsilon_{ij}$ on this $2 \times 2$ submatrix and set the other perturbation variables equal to zero. Since $E$ has the form (1), a system with five equations and four variables with rank 3 arises. Therefore, the four remaining perturbation elements only depend on 1 perturbation variable, say $x$. Now there are three possibilities for the directional derivative $D_x \Delta(A)$:

- If $D_x \Delta(A) = kx$, with $k > 0$, then choose $x > 0$. This is possible, since the sign of $x$ doesn’t matter for the matrix $A$ as long as $x$ is chosen small enough, $A$ is a stochastic matrix.
- If $D_x \Delta(A) = kx$, with $k < 0$, then choose $x < 0$ (for the same reason as the previous case).
- If $D_x \Delta(A) = 0$, then we consider $D_x^2 \Delta(A) = (\epsilon_{22}(v_1 + v_2) + \epsilon_{23}v_2 + \epsilon_{32}v_3 + \epsilon_{33}(v_1 + v_3))^2 + 4(\epsilon_{23}\epsilon_{32} - \epsilon_{22}\epsilon_{33})v_1(v_1 + v_2 + v_3)$. Since we have only 1 perturbation variable, it is easily verified that the second term is equal to zero. Therefore, only a nonzero quadratic term remains, which can be chosen positive. The sign of $x$ doesn’t matter, since for $x$ chosen small enough, the matrix-elements of $A$ are elements of the interval $[0, 1]$, because the $2 \times 2$ submatrix only consists of elements in the open interval $(0, 1)$.

Because of this lemma, we don’t have to investigate the cases of 0, 1 or 2 zero elements in the matrix $A$, since a non-diagonalizable stochastic $3 \times 3$ matrix with maximum 2 zero elements always contains a $2 \times 2$ submatrix with no zero element.

A non-diagonalizable stochastic $3 \times 3$ matrices $A$ satisfies (see [1]):

$$[\text{tr}(A) - 1]^2 - 4 \det(A) = 0$$

This equality is useful to determine whether a stochastic $3 \times 3$ matrix is non-diagonalizable. Another useful lemma in reducing the number of matrices to be examined is the following:

**Lemma 2.2.** If a perturbation matrix $E$ exists for a non-diagonalizable stochastic $3 \times 3$ matrix $A$, then there exists a perturbation matrix $F$ for every non-diagonalizable stochastic $3 \times 3$ matrix $TA^{-1}$, with $T$ a permutation matrix.

**Proof.** Consider $\tilde{A} = A + E$, which suffices the conditions (a)-(d). Then $T \tilde{A} T^{-1} = T(A + E)T^{-1} = TAT^{-1} + TET^{-1}$ suffices also the conditions (a)-(d):

(a) $T \tilde{A} T^{-1} = TAT^{-1} + TET^{-1}$ is an additive perturbation of $TAT^{-1}$.
(b) $T\tilde{A}T^{-1}$ is a stochastic matrix, since $\tilde{A}$, $T$ and $T^{-1}$ are stochastic matrices.

c) $T\tilde{A}T^{-1}$ has distinct eigenvalues, since $\tilde{A}$ has distinct eigenvalues and a similarity transformation preserves the eigenvalues.

d) $T\tilde{A}T^{-1}$ has the same principal left eigenvector as $TAT^{-1}$, since $\tilde{A}$ has the same principal left eigenvector as $A$.

3 Finding perturbations preserving stochasticity

Since a stochastic matrix has rowsums 1, a stochastic matrix with 7, 8 or 9 elements equal to zero doesn’t exist. Therefore we can disregard these cases and start with the case of 6 zero elements. In this paragraph, we denote the perturbation variable $x$, which we consider to be positive and sufficiently small. We also use parameters to describe the different cases. These parameter are $a$, $b$ and $c$, and they lie in the interval (0, 1).

3.1 6 zero elements

The only options for a stochastic matrix with 6 zero elements, are the matrices with a 1 and two 0’s on each row. The stochastic matrices which have 6 zero elements and are non-diagonalizable are:

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
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\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
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0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

Since these six matrices are all permutations of each other, it is sufficient to show that one of these matrices has a SCEP-perturbation such that it remains stochastic, according to lemma 2.2. The matrix

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

can be perturbed via the SCEP-algorithm into the matrix

\[
\tilde{A} = \begin{pmatrix}
0 & 1 - x & x \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

One can verify that $A$ and $\tilde{A}$ both have principal left eigenvector $(0,1,0)$ and $\tilde{A}$ has three distinct eigenvalues $1$, $\sqrt{x}$ and $-\sqrt{x}$. In this way, condition (a)-(d) are fulfilled.

3.2 5 zero elements

A stochastic $3 \times 3$ matrix with five zero elements must have two rows consisting of a 1 and two 0’s and a third row consisting of a zero, an element $b$ and an element $1-b$, with $0 < b < 1$. The non-diagonalizable stochastic $3 \times 3$ matrices are the following:

\[
\begin{pmatrix}
0 & b & 1-b \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & b & 1-b \\
0 & 1 & 0 \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
0 & b & 1-b \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
0 & b & 1-b \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & b & 1-b \\
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\begin{pmatrix}
1 & 0 & 0 \\
0 & b & 1-b \\
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\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
0 & b & 1-b \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
0 & b & 1-b \\
0 & 0 & 1
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
0 & 0 & 1 \\
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and

\[
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\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]
and their permutations. The first six matrices are all perturbations of each other, therefore only one of those matrices must be examined. If we consider

\[ B = \begin{pmatrix} 0 & b & 1-b \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \]

then

\[ \tilde{B} = \begin{pmatrix} x & b-x & 1-b \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \]

with \( 0 < x \ll b \), is a possible perturbation, resulting from the SCEP-algorithm. \( \tilde{B} \) has principal left eigenvector \((0, 0, 1)\), just as \( B \). And \( \tilde{B} \) has three distinct eigenvalues 1, 0 and \( x \).

The matrices

\[ C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 3/4 & 1/4 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/4 & 3/4 & 0 \end{pmatrix} \]

can be perturbed into

\[ \tilde{C} = \begin{pmatrix} x & 0 & 1-x \\ 1-4x & 0 & 4x \\ 3/4 & 1/4 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{D} = \begin{pmatrix} 4x & 1-4x & 0 \\ 0 & 0 & 1 \\ 1/4-x & 3/4+x & 0 \end{pmatrix} \]

### 3.3 4 zero elements

The stochastic matrices with 4 zero elements consist of two rows with a zero and two terms which add to 1 and a row with a 1 and two 0’s. The matrices which have a \( 2 \times 2 \) submatrix without 0’s do not have to be examined because we know that they can be perturbed in the proposed manner, according to lemma 2.1. In this case, the non-diagonalizable variants of these matrices are:

\[ A = \begin{pmatrix} a & 1-a & 0 \\ b & 0 & 1-b \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} a & 1-a & 0 \\ 0 & b & 1-b \\ 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} a & 1-a & 0 \\ 0 & b & 1-b \\ 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} a & 1-a \\ b & 1-b & 0 \\ 0 & 1 & 0 \end{pmatrix} \]

\[ G = \begin{pmatrix} a & 1-a \\ 0 & 0 & 1-b \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} a & 1-a \\ b & 0 & 1-b \\ 0 & 1 & 0 \end{pmatrix} \]

The matrix \( A \) is only be non-diagonalizable if \( a = 4b - 3 \). Therefore, we consider the matrix

\[ A' = \begin{pmatrix} 4b-3 & 4-4b & 0 \\ b & 0 & 1-b \\ 1 & 0 & 0 \end{pmatrix}. \]

\( A' \) can be perturbed by the SCEP-algorithm into the stochastic matrix:

\[ \tilde{A}' = \begin{pmatrix} 4b-3-4(1-b)x & 4-4b & 4(1-b)x \\ b+x & 0 & 1-b-x \\ 1 & 0 & 0 \end{pmatrix}. \]

The matrix \( B \) is only non-diagonalizable if \( b = 1-a + 2\sqrt{1-a} \). Therefore, we consider the matrix

\[ B' = \begin{pmatrix} a & 1-a \\ 0 & a-1+2\sqrt{1-a} \\ 1 & 0 & 2-a-2\sqrt{1-a} \end{pmatrix}. \]
$B'$ can be perturbed by the SCEP-algorithm into the stochastic matrix

$$
\tilde{B}' = \begin{pmatrix}
a & 1 - a & 0 \\
0 & a & 1 - a \\
0 & 0 & 1
\end{pmatrix}.
$$

The matrix $C$ is only non-diagonalizable if $a = b$. Therefore, we consider the matrix

$$
C' = \begin{pmatrix}
a & 1 - a & 0 \\
0 & a & 1 - a \\
0 & 0 & 1
\end{pmatrix}.
$$

$C'$ can be perturbed by the SCEP-algorithm into the stochastic matrix

$$
\tilde{C}' = \begin{pmatrix}
a - x & 1 - a & x \\
x & b & 1 - b - x \\
0 & 0 & 1
\end{pmatrix}.
$$

The matrix $D$ is only non-diagonalizable if $b = 4 - 4a$. Therefore, we consider the matrix

$$
D' = \begin{pmatrix}
0 & a & 1 - a \\
4 - 4a & 4a - 3 & 0 \\
0 & 1 & 0
\end{pmatrix}.
$$

$D'$ can be perturbed by the SCEP-algorithm into the stochastic matrix

$$
\tilde{D}' = \begin{pmatrix}
0 & a & 1 - a \\
4 - 4a - 4(1 - a)^2x & 4a - 3 + 4(1 - a)^2x & 0 \\
x & 1 - x & 0
\end{pmatrix}.
$$

The matrix $G$ is only non-diagonalizable if $a = \frac{1}{4-4b}$. Therefore, we consider the matrix

$$
G' = \begin{pmatrix}
0 & \frac{1}{4-4b} & \frac{3-4b}{4-4b} \\
b & 0 & 1 - b \\
1 & 0 & 0
\end{pmatrix}.
$$

$G'$ can be perturbed by the SCEP-algorithm into the stochastic matrix

$$
\tilde{G}' = \begin{pmatrix}
0 & a - x & \frac{1 - a + x}{(4 - 4b)x} \\
\frac{1}{4-4a} & \frac{3-4a}{4-4a} & \frac{3 - 4a}{4-4a} - (4 - 4b)x \\
1 & 0 & 0
\end{pmatrix}.
$$

The matrix $H$ is only non-diagonalizable if $b = \frac{1}{4-4a}$. Therefore, we consider the matrix

$$
H' = \begin{pmatrix}
0 & a & 1 - a \\
\frac{1}{4-4a} & 0 & \frac{3-4a}{4-4a} \\
0 & 1 & 0
\end{pmatrix}.
$$

$H'$ can be perturbed by the SCEP-algorithm into the stochastic matrix

$$
\tilde{H}' = \begin{pmatrix}
0 & a + (4 - 5a)x & 1 - a - (4 - 5a)x \\
\frac{1}{4-4a} & 0 & \frac{3-4a}{4-4a} \\
0 & 1 - x & \frac{3 - 4a}{4-4a}
\end{pmatrix}.
$$
### 3.4 3 zero elements

In the case of stochastic $3 \times 3$ matrices with 3 zero elements, with the help of lemma 2.1, only two possibilities remain, namely:

$$A = \begin{pmatrix} a & 1-a & 0 \\ b & 0 & 1-b \\ 0 & c & 1-c \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1-a & a \\ b & 0 & 1-b \\ c & 1-c & 0 \end{pmatrix}$$

The stochastic matrix $A$ is never non-diagonalizable, since $[\text{tr}(A) - 1]^2 - 4 \det(A) > 0$. It is possible to perturb the stochastic matrix $B$ via the SCEP-algorithm, if $K > 0$, $L > 0$, $M > 0$ or $N > 0$. If one of these inequalities holds, then we only perturb on the corresponding perturbation element (according to (2)) with a positive perturbation. This results in a positive perturbation on $b_{11} = 0$, and thus $b_{11} > 0$. For example, if $L > 0$, then we perturb on $b_{23}$ with $\epsilon_{23} > 0$. This leads to a negative perturbation on $b_{13}$ and $b_{21}$ and a positive perturbation on $b_{11}$.

The case that $K < 0$, $L < 0$, $M < 0$ and $N < 0$ does not happen: The matrix $B$ is only non-diagonalizable if $1 - a - b \neq 0$ and $c = \frac{\frac{1}{4} - ab}{1-a-b}$. For $\frac{\frac{1}{4} - ab}{1-a-b}$ to be an element of the interval $[0,1]$, the pair $(a,b)$ is restricted to the region where $a > \frac{1}{4b}$ or $a < \frac{3-4b}{4-4b}$, the solutions $(a,b)$ are represented by the regions $G$ and $H$ in figure 1. The blue lines represent the pairs $(a,b)$ where $L = 0$, within those blue lines, we have that $L < 0$. It is clear from figure 1 that for every pair $(a,b)$ with $a > \frac{1}{4b}$ or $a < \frac{3-4b}{4-4b}$, we have that $L > 0$. Since $L$ is always positive in this case, it is possible to determine a perturbation with $\epsilon_{23} > 0$ and $\epsilon_{22} = \epsilon_{32} = \epsilon_{33} = 0$ such that there exists an $E$ that suffices the conditions (a)-(d).

### 3.5 Conclusion

The paper shows for all possible cases (some explicitly and some via lemma 2.1) that there exists a perturbation matrix $E$ such that the conditions (a)-(d) are sufficed. From this, we can conclude that for every zero-element $a_{ij} = 0$ in a stochastic matrix, there exists a perturbation matrix $E$ with $\epsilon_{ij} > 0$ such that the conditions (a)-(d) are sufficed.
References