Reference Matrix for Multi-Axis Random Vibration Control Tests

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Declaration of Authorship

I, Giacomo D’ELIA, declare that this thesis titled, “Reference Matrix for Multi-Axis Random Vibration Control Tests” and the work presented in it are my own. I confirm that:

- This work was done wholly while in candidature for a research degree at this University;
- This work has been supervised and approved by Professor Emiliano Mucchi, Professor Giorgio Dalpiaz, Eng. Bart Peeters and Eng. Umberto Musella;
- This work is part of the paper Analyses of target definition processes for MIMO Random Vibration Control Tests submitted and approved for the IMAC XXXV conference, Los Angeles (USA), 2017;
- This work is part of the paper Tackling the target matrix definition in MIMO Random Vibration Control testing submitted and approved for the ATS conference, Los Angeles (USA), 2017;
- The work has been done mainly at Siemens Software Industries N.V. (Interleuvenlaan 68, 3000 Leuven, Belgium);
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Abstract

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Reference Matrix for Multi-Axis Random Vibration Control Tests

by Giacomo D’ELIA

Nowadays the advantages of performing Multiple-Input Multiple-Output (MIMO) Random Vibration Control tests are widely accepted by the environmental engineering community. However their practice still needs to grow because of the high degree of expertise needed to perform these tests. The challenges of MIMO Random Control start even before the actual test, in the test definition phase. The target that needs to be reached during the test is a full Spectral Density Matrix. Defining this matrix with no a-priori knowledge of the cross-correlation between control channels could be problematic: filling in the off-diagonal terms, in fact, must guarantee that the target has a physical meaning. This is translated in the algebraic constraint that the target matrix needs to be positive semi-definite. On the other hand the pushing driver of any Random Vibration Control test is to be able to replicate specific PSDs (test requirements). Even if several author tackled the problem to define a realisable full MIMO Random Reference, a clear gap in the standard procedures about a generally accepted and robust method still exists. The purpose of this work is to investigate more innovative target generation procedures pointing out the advantages and the challenges in terms of physical meaning and their impact on the random control strategy, in order to aim to a well-defined automatic procedure to be included in the standard practice.
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List of Abbreviations

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<td>Multiple-Input-Multiple-Output</td>
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<td>VibCo</td>
<td>Vibration Control</td>
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<td>PSD</td>
<td>Power Spectral Density</td>
</tr>
<tr>
<td>CSD</td>
<td>Cross Spectral Density</td>
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<tr>
<td>SDM</td>
<td>Spectral Density Matrix</td>
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<tr>
<td>MDOF</td>
<td>Multiple Degree Of Freedom</td>
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<tr>
<td>LTI</td>
<td>Linear Time Invariant</td>
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<tr>
<td>FRF</td>
<td>Frequency Response Function</td>
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<tr>
<td>DFT</td>
<td>Discrete Fourier Transform</td>
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<td>SVD</td>
<td>Singular Value Decomposition</td>
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Chapter 1

Introduction

Multi-axis vibration control tests are performed to subject a unit under test to a realistic three-dimensional dynamic environment. Nowadays the common practice when multiple directions are required for testing the exposure of a structure to a given vibration environment, is to perform sequential single-axis tests (fig.1.1). It has been shown in numerous publications that single-axis vibration tests poorly represent the stresses of the naturally multi-axial operational environment. As explained by Peeters in [1] and Musella in [2], this practice has known drawbacks: first of all, there is a lack of physical meaning in replicating in-service conditions with a single axis test due to the fact that the in-service stress loading is typically multi-axial; furthermore, sequential single-axis applications can be time consuming and dangerous, since the test set-up has to be changed multiple times with the additional unacceptable risk of damaging the structure. In case of testing heavy slender structures to high overall levels, a single input could also lead to high concentrated loads with the consequence of entering the non-elastic range of the material properties. As well for testing of large structures, it is possible that the required force to be input is not available from a single shaker.

Multiple-Input Multiple-Output (MIMO) control strategy overcomes these limitations and substantial benefits can be gained on using it, as proved in [3]. Although the benefits of multi-axis vibration control tests (fig.1.2) are clear and accepted, the current industrial practice is still to a very large extent relying on Single-Input Single-Output (SISO) control. There is often demand to carry out testing in a very similar manner to historical tests simply for continuity, or due to the high degree of expertise needed to perform MIMO tests [4]. However, the recent updates to include tailoring guidelines for multi-exciter testing in the United States Military Standards [5], highlight
the necessity to improve the knowledge of this innovative practice.

1.1 Target generation problems for multi-axis vibration control tests

There are several types of Multi-Axial Vibration Control qualification tests (i) MIMO Random Test, (ii) MIMO TimeWaveform Replication and (iii) MIMO Sine Control Test, differing from the nature of the excitation environment the specimen needs to be subjected to. The focus of this work is only on Random VibCo tests, which are used to simulate the response of the unit under test to a random vibration environment; typical scenarios are the road excitation or the payload and avionic equipment responses during a spacecraft launch.

In random vibration qualification testing, the qualification requirements are given in terms of Power Spectral Density (PSD) that need to be reproduced at a certain control channel location. In the MIMO case, the qualification requirement cannot be a single PSD anymore, therefore additional information about the cross-talk between these channels need to be included. This information must be provided in terms of Cross Spectral Densities (CSDs) between the control channels defining, for instance, desired phases and coherences.

The target for a multi-exciter random control test is thus a full reference Spectral Density Matrix (SDM). Defining then the full MIMO reference matrix with no a-priori knowledge of the cross-correlation between control channels is very challenging. Filling in the off-diagonal terms, in fact, must guarantee that the reference matrix has a physical meaning. This is translated in the algebraic constraint that this matrix needs to be positive semi-definite, without neglecting the pushing constraint that the test has still to guarantee the required levels at the control locations (ch. 3.1). In this work, it will be called "mathematically realisable" a positive semi-definite SDM. The solution of a mathematically realisable reference matrix with fixed PSD terms is not unique and not all the possible solutions can be exactly reproduced in the laboratory for a given specimen. In this sense, it will be called "physically realisable" a SDM that respects all the physical
1.2 State of the art and purpose of the work

In multi-axis vibration control tests, the generation of the reference matrix with a physical meaning (mathematically and physically realisable) is a crucial point. The best information an environmental test engineer can use is a target originated from real life recordings or field data. However, even in this case, when trying to replicate the target on the test rig, there will be inevitable impedance mismatches between the infield and the laboratory conditions. There is thus always the risk of trying to force the test item to an unnatural motion that could lead to poor control results and performances [3],[6],[7]. A target with high control performances can then be achieved by minimizing the aforementioned impedance mismatch on top of using field data (IMMAT procedure, [3],[6]).

Other considerations can also drive the choice of a reference matrix. For instance a desirable condition could be to minimize (maximize) the energy provided to the shakers to reach the given target levels. A reference that minimizes (maximizes) the input power is said to match the Minimum (Maximum) Extrime Drives Requirements [5],[8] and in the following will be addressed as Extreme Drives Target. Minimizing required drives power to reach fixed output levels is an attractive solution that can increase the reachable levels during the test, avoiding the data acquisition system drives overload and preserving the exciters nominal power limits (that sometimes are very limiting thresholds for MIMO Random Control tests). This problem has been tackled by Smallwood in [8], Musella in [2][9] and the method is also mentioned in the standard practice for multi-exciter vibration tests [5].

Even if several authors tackled the problem of defining the best target possible in terms of minimum drives energy [8], in terms of control performances [3],[6],[7], few works can be addressed that tackle the problem of defining a physically realizable target first [1][2][9]. This leads to a clear gap in the procedures about a generally accepted and robust method to define a full MIMO Random reference matrix.

The purpose of this work is to investigate different state-of-art and constraints related to the sensor configuration used for the test.
more innovative target generation procedures, pointing out the advantages and the challenges in terms of physical meaning and their impact on the random control strategy, in order to aim a well-defined automatic procedure to include in the standard practice.

Figure 1.1 – Single exciter vertical axis test setup [5]

Figure 1.2 – Tri-axial exciter test setup [5]
Chapter 2

MIMO Random Vibration Control Theory

In order to better understand the key role of the multi-axial test’s target definition, it is necessary to review some MIMO system theory. This chapter also clarifies the notation used throughout this master thesis. As explained in the following, the MIMO field is characterised by the use of matrices. The size of the matrix is typically stated an $[n,m]$, where $n$ is the number of rows and $m$ is the number of columns. In this work, three dimensional matrices are also used where the third dimension is typically samples in either the time or frequency domain. It is assumed that if the matrix has three dimensions, that the operations can be performed on each two dimensional matrix along the third dimension. For example if the matrix is a matrix of frequency response functions, matrix operations will be performed at each frequency line. In this work most of the derivations are in the frequency domain hence all the arrays are functions of the frequency $f$ (in Hz), if not specified otherwise.

Vectors are denoted by lower case bold letters, e.g. $\mathbf{a}$, and matrices by upper case bold letters, e.g. $\mathbf{A}$. An over-bar $\bar{\mathbf{a}}$ is used to indicate the complex conjugate operation and the apex $\mathbf{A}'$ to indicate the complex conjugate transpose of a matrix, e.g. $\bar{\mathbf{a}}$ and $\mathbf{A}'$ are the complex conjugate and the complex conjugate transpose of the vector $\mathbf{a}$ and the matrix $\mathbf{A}$, respectively. The dagger symbol $\mathbf{A}^\dagger$ is used to indicate the Moore-Penrose pseudo-inverse of a matrix, whereas the hat $\hat{\mathbf{A}}$ is used to emphasize the estimation of a quantity, e.g. $\hat{\mathbf{A}}$ is the estimate of the matrix $\mathbf{A}$. 


2.1 Frequency Domain Transfer Function Relationship

Multiple-Input-Multiple-Output (MIMO) refers to input of multiple drive signals to an exciter system in a MDOF configuration, and multiple measured outputs from the fixture or test item in a MDOF configuration [5].

A general MIMO linear system is represented in fig. 2.1. The \( m \) inputs and the \( \ell \) outputs can be represented as two vectors, \( u(t) = \{u_1(t), ..., u_m(t)\}^T \) and \( y(t) = \{y_1(t), ..., y_\ell(t)\}^T \), respectively. It is important to note that generally, there is no one to one correspondence between inputs and outputs, and the number of inputs and the number of outputs may be different \( (\ell \geq m) \).

![Figure 2.1 – MIMO linear system](image)

Each element in the vectors is a time history, and the elements in the vectors can also be represented in the frequency domain by the Fourier Transform [10]

\[
U_i(f) = \int_{-\infty}^{+\infty} u_i(t) e^{-i2\pi ft} \, df \quad i = 1, ..., m \tag{2.1}
\]

\[
Y_i(f) = \int_{-\infty}^{+\infty} y_i(t) e^{-i2\pi ft} \, df \quad i = 1, ..., \ell \tag{2.2}
\]

This integral does not strictly exist for a stationary random signal and a limiting operation must be defined. It is thus possible to define \( U(f) \in \mathbb{C}^{m \times 1} \) as column vector of the \( m \) inputs signals and \( Y(f) \in \mathbb{C}^{\ell \times 1} \) as a column vector of the \( \ell \) outputs signals

\[
U = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix} \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_\ell \end{bmatrix}
\]
2.1. Frequency Domain Transfer Function Relationship

In the hypothesis of a Linear Time-Invariant (LTI) system, the structure under test is represented by the Frequency Response Function (FRF) matrix, \( H \in \mathbb{C}^{\ell \times m} \)

\[
Y = HU \tag{2.3}
\]

Multiplying by \( U' \) both side of the eq. 2.3, the input-output relationship can be rewritten as

\[
YU' = HUU' \tag{2.4}
\]

where by performing an averaging operation \([10]\]

\[
S_{uu} = E[UU'] \quad \text{Input Spectral Density Matrix} \tag{2.5}
\]
\[
S_{yy} = E[YY'] \quad \text{Output Spectral Density Matrix} \tag{2.6}
\]
\[
S_{yu} = E[YU'] \quad \text{Cross Spectral Density Matrix} \tag{2.7}
\]

An unbiased estimate of the spectrum is the so-called weighted periodogram: the time series is weighted by a window to reduce leakage (e.g. a Hanning window), the discrete Fourier transform (DFT) is applied to the weighted time series and the DFT is multiplied by its complex conjugate transpose. Typically, the variance of the estimate reduces by splitting the original time signals in (overlapping) segments, computing the weighted periodograms of all segments and taking the average.

Because of a column vector is multiplied by a row vector, a periodogram is a rank-one estimate of the spectrum. Segment averaging increases the rank. Care must be taken that the number of averages is at least equal to the dimension of the estimated spectral density matrix \([1]\).

By substituting the definitions of Auto and Cross power in the eq. 2.4

\[
S_{yu} = H S_{uu} \tag{2.8}
\]

\((\ell \times m) \quad (\ell \times m)(m \times m)\)

As explained in the following (chapter 2.3), in performing laboratory multi-axis tests, the initial estimation of \( H \) will be computed based on a set of uncorrelated random input signals, performing the System
Identification. The so-called $H_1$ estimator is

$$\hat{H} = \hat{S}_{yu} \hat{S}_{uu}^\dagger$$ \hspace{1cm} (2.9)

where $\hat{S}_{yu}$ and $\hat{S}_{uu}^\dagger$ are the spectral density matrices estimated, for instance, via the Welch’s Averaged Periodogram.

### 2.2 From Outputs to Inputs

A simplified version of a MIMO random control test can be generalized in fig. 2.2. The objective of the control system is to create a drive signal vector, $u(t)$, which causes the Output Spectral Density Matrix, $S_{yy}$, to agree, within some acceptable error margin, with the user specified Reference Spectral Density Matrix, named $S_{yy}^{ref}$.

![Figure 2.2 – Basic representation of a control system](image)

Theoretically speaking, starting from the user specified Reference SDM ($S_{yy}^{ref}$), it is always possible to determine the Input SDM ($S_{uu}$) (inverse control problem [5])

$$Y = HU \quad \Rightarrow \quad U = H^\dagger Y$$

$$U' = (H^\dagger Y)' = Y'H'^\dagger$$

$$UU' = H^\dagger YY'H'^\dagger$$

$$S_{uu} = H^\dagger S_{yy}^{ref} H'^\dagger$$ \hspace{1cm} (2.10)

Eq. 2.10 shows as in MIMO control, in order to theoretically define the output-to-input relationship, the matrix-inverse of the FRF is needed. Note that for the general case in which the number of
the input drive signals ($m$) and the number of the output control response signals ($\ell$) is not the same, the FRF will be rectangular ($H \in \mathbb{C}^{\ell \times m}$) and the computation of the inverse FRF ($H^{-1}$) will require a pseudo-inverse approximation. The most used is the Moore-Penrose pseudo-inverse. It can be computed by the use of a procedure closely related to the singular value decomposition (SVD), which provides the matrix condition number at each frequency and easily allows the system to properly control in the least-squares sense. The Moore-Penrose pseudo inverse of the FRF can be derived as follows [5]

$$H = TSV'$$
$$T'H = T'SV' = SV'$$
$$S^{-1}T'H = S^{-1}SV' = V'$$
$$VS^{-1}T'H = VV' = I$$
$$H^\dagger = VS^{-1}T'$$ (2.11)

where $T$ and $V'$ are orthonormal and $S$ is a diagonal matrix of non-negative real numbers. To compute $S^{-1}$ the elements greater than a tolerance are inverted and kept, the elements less than a tolerance are replaced by zero. The SVD decomposition, first provides information on a spectral line basis as to the rank of $H$, and second it provides the methods of addressing dynamic range and noise by investigating the ratio of the largest to smallest singular value.

The pseudo inverse of the FRF ($H^\dagger$) is often called the Impedance matrix of the unit under test, ($Z$); therefore the output-to-input relationship (eq. 2.10) can be rewritten as

$$S_{uu} = \hat{Z} \quad S_{yy}^{\text{ref}} \quad \hat{Z}'$$

$$\left(m \times m\right) \quad \left(m \times \ell\right) \left(\ell \times \ell\right) \left(\ell \times m\right)$$ (2.12)

As discussed in the previous section, the impedance needs to be estimated prior to the test in the System Identification phase. This matrix inversion has to be performed with care since singularities can exit at various frequencies. Nonlinear response and measurement noise can also seriously affect the accurate determination of the impedance, having a negative impact on the quality of control. The Moore-Penrose pseudo-inverse, if evaluated correctly, can handle many of these cases.


2.3 MIMO Random Control Algorithm

The purpose of a MIMO random vibration control test is to reproduce certain user-defined spectra at specific output locations. In other words, the objective of the test is to generate the input signals vector \( u(t) \) that, driving the shakers, is able to replicate the full Reference SDM, \( S_{yy}^{ref} \).

As explained before, theoretically the reference matrix could be directly achieved by sending the input drives that have the specified Input SDM:

\[
u(t) = \text{ifft}(U) : S_{uu} = E[UU'] = \hat{Z}S_{yy}^{ref}\hat{Z}'
\]  

(2.13)

Nevertheless, due to the possible non-linear behaviour of the unit under test and noise in the measurements, the system estimated in the pre-test phase will inevitably differ from the actual one \((HZ' \neq I)\):

\[
S_{yy} = (HZ\hat{Z})S_{yy}^{ref}(HZ\hat{Z})' \neq S_{yy}^{ref}
\]  

(2.14)

and a control action is needed to reduce the error

\[
E = S_{yy}^{ref} - S_{yy}
\]  

(2.15)

Nowadays the various MIMO control system vendors do not always approach control in the same manner and some innovative random vibration control strategies have been proposed by few authors, Smallwood in [11][12], Peeters in [1] and Underwood in [13].

It is not the intent of this master thesis to provide the specifics of the control algorithms used in the conduct of multi-axis tests, however there are a few basic concepts that are keys to the MIMO control problem that will be addressed in the following. MIMO random vibration control algorithm is represented in the flow chart in fig. 2.3.

During a test, as explained above, non-linearities and noise make impossible the use of the theoretically computed input drives. To overcome this problem, the so called Cholesky Factorisation is applied directly to the Reference SDM. The Cholesky Factorisation decomposes a positive semi-definite matrix in a lower triangular matrix times its complex conjugate:

\[
S_{yy}^{ref} = LL'
\]  

(2.16)
Using eq. 2.12 and eq. 2.16, it has been demonstrated [12] that the so-called Decoupled Inputs can be computed as

\[ \mathbf{U}^{\text{DC}} = \hat{\mathbf{Z}} \mathbf{L} \mathbf{W} \]  

(2.17)

where \( \hat{\mathbf{Z}} \) is the Impedance of system, estimated during the System Identification phase with a low-level vibration pre-test; \( \mathbf{L} \) is the Cholesky factor of the Reference matrix, that is a lower triangular complex matrix with real and non-negative diagonal elements; \( \mathbf{W} \) represents the Multisine Matrix of which all sine components have unit amplitude and random phase. This matrix is an approximation of white noise, and its elements are called pseudo-random signals [1].

The decoupled input signals are converted in the time domain by applying the inverse DFT. The matrix formulation of the problem makes possible the application of a Time-Domain Randomisation. The randomisation procedure converts the line spectrum from the pseudo-random signals to a true continuous spectrum and ensures a continuous input data stream to the shaker, without having to carry out intensive matrix computations [14]. To meet these requirements, the time-domain randomisation procedure outlined by Smallwood in [12] is used. After the randomisation, the input drive signals \( \mathbf{u}(t) \) can finally be send to the shakers and, exciting the structure, the output signals \( \mathbf{y}(t) \) can be measured. Once the Output SDM (\( S_{yy} \)) is estimated, the Error Correction (\( \Delta \)) can be computed. The error is on
the lower triangular matrix $L$ based on the difference between the Reference SDM ($S_{yy}^{ref}$) and the Output SDM ($S_{yy}$). The error correction procedure is outlined by Smallwood in [12] and Peeters in [14]. The complexity of building these systems, e.g. designing the control system and specifying the test parameters, increases much faster than the rate of increase in the number of actuators. To a first order, the control and test specification complexity increases by at least the square of the number of actuators that are used [15].

It is also important to emphasise the critical role of the system identification to the success of a MIMO control test. The estimate of the impedance matrix is needed to generate the decoupled inputs (eq. 2.17), and considering that it is only an estimate, it may have some error at certain problem frequencies. These frequencies correspond to structural anti-resonances, that are frequencies at which the system is either under or overdetermined, or frequencies at which the system-under-test responds inefficiently to the excitation energy, due to the low-response behaviour. The frequencies at which the magnitude of the elements of $H$ are too small, its columns or rows are linearly dependent, near-singularities or singularities occur and the measurements contains the most error [15].

A typical MIMO FRF matrix (3x3 matrix of bode plots), estimated during the system identification phase, is represented in fig. 2.4 [16]. Each column of the figure is relative to one drive, and each row to one control channel. A heavy anti-resonance is highlighted at 1180 Hz and critical conditions in the control behaviour can arise because of the drives level required may be considerably high, as shows fig. 2.5 [16]. To narrow these problems, the system characterization is done by exciting all actuators in the system-under-test simultaneously with band-limited Gaussian noise. The drive signals are typically flat spectrally and band limited to the maximum frequency of interest. They are also uncorrelated among themselves. The response levels for the system characterization should be chosen as high above the noise floor as possible to maximize the accuracy of the impedance estimate, but still ensure that the test article does not experience vibration levels above some operator chosen level, in order to not excessively stress the test article.

However good mechanical design (e.g. the design of the actuation systems, the test fixture subsystems and the control-points set-up) is
very important and can reduce the severity of system identification and control problems that can arise during a MIMO random control test.

**Figure 2.4** – 3x3 RFR matrix from the Sys. Id. [16]

**Figure 2.5** – Top: Required drive power spectral density; Bottom: Estimated singular values. [16]
Chapter 3

Reference Spectral Density Matrix

In vibration control, the user specifies the Reference Spectral Density Matrix and the control algorithm has to produce input time series that drive the shakers. In MIMO Random Control not only the vibrations at each user specified control location need to be verify, but also the vibration relations that exist between each set of control points. In this sense, the target that needs to be reached during the test must to be a full Spectral Density Matrix where the cross terms are as important as the diagonal ones. Therefore the challenges of MIMO Random Control start even before the actual test, in the test definition phase. Filling in the reference matrix with no a-priori knowledge of the cross-correlation between control channels in fact, could not guarantee that the matrix has a physical meaning. In order to be sure that the reference matrix is realisable and thus usable inside the control algorithm, it must to be a positive semi-definite matrix. Starting from this fundamental algebraic constraint, this chapter will discuss different state-of-art and more innovative target generation procedures.

3.1 Features of the Reference Matrix

A schematic MIMO system is represented in fig. 3.1; two shakers excite the structure and the responses are recorded in the three user defined control accelerometers. In this basic example, a 3 by 3 Reference Spectral Density Matrix needs to be defined, putting the required Power Spectral Densities (PSDs) as diagonal terms and the Cross Spectral Densities (CSDs) as off-diagonal terms (fig. 3.2).
Fig. 3.3 shows a more general reference matrix with \( \ell \) user defined control channels. The matrix is a 3 dimensional matrix and at each frequency line (the 3rd index) it is a square complex matrix. Also, note the Hermitian structure due to the fact that the diagonal elements are real positive numbers (PSDs) and the corresponding off-diagonal elements are complex conjugate pairs.

One of the most important aspects on defining the Reference SDM is that it must be a positive semi-definite matrix [5]. This fundamental algebraic constraint, first of all guarantees the physical meaning of the reference and also leads to a realisable target for the control test. As explained in ch. 2.3 in fact, the first step in the MIMO random control algorithm is the Cholesky factorisation of the Reference SDM (eq.2.16). This operation can only be applied to a positive semi-definite matrix therefore this mathematical constraint can not be left out.

In linear algebra, a matrix \( A \in \mathbb{R}^{n \times n} \) is positive semi definite if \( x^T A x \geq 0 \) for all non-zero \( x \in \mathbb{R}^n \) [17].

Beside the algebraic definition, there are some important properties
to consider for practical applications. Particularly, the following statements are equivalent to the Reference SDM \( S_{yy}^{\text{ref}} \) being positive semi-definite [17][18]:

a) all the eigenvalues of \( S_{yy}^{\text{ref}} \) are semi-positive;

b) \( S_{yy}^{\text{ref}} \) has a unique Cholesky Decomposition, meaning that it can be decomposed in the product of two triangular hermitian matrices, referred as the Cholesky Factors: \( S_{yy}^{\text{ref}} = LL' \);

c) the Sylvester’s Criterion is respected, i.e. all the principal minors of \( S_{yy}^{\text{ref}} \) have positive determinants. The principal minors are the square sub-matrices that share the diagonal with the full matrix.

Moreover, if \( S_{yy}^{\text{ref}} \) is positive semi-definite then:

d) the trace of \( S_{yy}^{\text{ref}} \) is real and semi-positive, being the matrix trace the sum of its eigenvalues;

e) the determinant of \( S_{yy}^{\text{ref}} \) is real and semi-positive, being the matrix determinant the product of its eigenvalues;

f) \( S_{uu} = ZS_{yy}^{\text{ref}}Z' \) is positive semi-definite.

\[ \begin{array}{cccc}
\text{PSD}_1 & \text{CSD}_{12} & \ldots & \text{CSD}_{1l} \\
\text{CSD}^*_{12} & \text{PSD}_2 & \ldots & \text{CSD}_{2l} \\
\vdots & \vdots & \ddots & \vdots \\
\text{CSD}^*_{1l} & \text{CSD}^*_{2l} & \ldots & \text{PSD}_l \\
\end{array} \]

**Figure 3.3** – Reference Spectral Density Matrix for \( \ell \) control points

### 3.2 Building the Reference Matrix

In the conduct of a MIMO Random VibCo test, as one would expect, the particular configuration of the channel set-up and the definition of the control point locations on the test item will influence
the reference spectral requirements. Both Power Spectral Density and Cross-Spectral Density terms are in fact required test parameters. Even if the diagonal terms of $S_{yy}^{\text{ref}}$ are usually known levels for the environmental test engineer, provided as test specifications, the off-diagonals terms (CSDs in fig. 3.3) are often unknown quantities and need to be defined.

The cross-terms can be determined from the respective PSDs; for instance, the Cross-Spectral Density between two control channels can be computed via

$$CSD_{ij} = |CSD_{ij}| e^{i\phi_{ij}} = \gamma_{ij} \sqrt{PSD_i PSD_j} e^{i\phi_{ij}}$$  \hspace{1cm} (3.1)$$

where $i$ and $j$ are the $i$-th and the $j$-th control channels and $\gamma_{ij}$ and $\phi_{ij}$ are the coherence value and the phase angle between the two control channels, respectively.

In the MIMO case, care must be taken in interpretation of coherence, because of its definition is closely related to eq. 3.1. The coherence function is defined as the correlation coefficient describing the linear relationship between any two single spectra

$$\gamma_{ij}^2 = \frac{|CSD_{ij}|^2}{PSD_i PSD_j}$$  \hspace{1cm} (3.2)$$

so that

$$0 \leq \gamma_{ij}^2 \leq 1$$  \hspace{1cm} (3.3)$$

In order to understand the meaning of coherence and phase of a cross-term a practical example is given. Considering two control channels on $x$ and $y$ directions, if the phase between the control points is equal to zero ($\phi_{xy} = 0$), the movement will preferably occur along a $45^\circ$ straight line between the normalized $x$ and $y$ axes. Moreover if the coherence is one ($\gamma_{xy} = 1$), it will be a perfect straight line, otherwise the movement will somewhat deviate from the straight line [1].

Using eq. 3.1 to define the full Reference SDM, only the PSD levels at each control points and the respective profile of coherence and phase between them are required. Most of the MIMO VibCo software in fact, have the possibility of defining element-wise the CSDs in terms of coherence and phase profiles. Fig. 3.4 shows the definition of a 3x3 Reference SDM in the Multi-axis Random Setup phase on using
3.2. Building the Reference Matrix

MIMO Random of Test.Lab, the Siemens LMS vibration control software. In this example the user has to specify the required three PSD profiles as diagonal terms and the desired profiles of coherence and phase for the upper off-diagonal elements (Hermitian matrix, see ch. 3.1). The control system computes the Cross-Spectral Densities and thus defines the full Reference SDM. It is also important to underlain that, for computation reason linked to the control process stability [7], coherence values of 0 and 1 are usually avoided. Typical values of low coherence and high coherence are $0.05 \div 0.08$ and $0.95 \div 0.98$, respectively.

![Reference Profiles](image)

**Figure 3.4** – Reference matrix definition phase on Test.lab MIMO Random Control

In the general case of $\ell$ user defined control channels (fig.3.3), filling in the MIMO reference matrix element by element in terms of the $\frac{\ell(\ell+1)}{2}$ coherence and phase profiles could result in $S_{yy}^{ref}$ being non positive semi-definite, following the fact that none of the properties (a) b) c) in ch. 3.1) have been taken into account in the completion process. The definition of valid coherence and phase profiles, resulting on a realisable full Reference SDM, is thus the main challenge. One of the most reliable way is taking measured field data profiles, where the measurements are originated from a real-life experiment under representative loading conditions [3] [6] [7]. Even if this method
seems to be easy, several problems cannot be underrated [5]. First, the boundary conditions in the field may be different from the boundary conditions in the laboratory. Second, the item on which the field data were taken may not be identical to the test item. Third, laboratory control locations may not be perfectly the same as the original measurement points. Fourth, the specification may be a composite of several environments, making the definition of cross-spectra very difficult. Small changes in the modal frequencies, caused by any of the above factors, can change the phase at any frequency near a mode by a large amount. There is thus always the risk of trying to force the test item to an unnatural motion that could lead to poor control results and performance.

When only specification PSD levels are known and measurement data of coherence and phase are not available or not correctly usable, the choice of providing the best solution is on the environmental test engineer.

Nowadays, to fill in the MIMO matrix being a realisable target, the standard procedures are [5]:

- assume that excitation environments are independent one from another. It means setting low coherence profiles through the frequency band of interest. For such a small coherence, the CSD terms are irrelevant and the phase parameter is essentially a random variable. In this case a phase specification is not required.

- assume that the responses are in phase ($\phi = 0$ for all the CSDs) and fully correlated.

These two methods, suggested by the United State Military Standards [5], can safely be used but they are clearly not general since there might be the necessity of replicating partially or fully correlated out of phase responses. This is the case, for instance, of a Multi-Axial shaking table that needs to be used to reproduce pure roll (rotation on x-axis), using two control channels on y-direction, as shown in fig. 3.5. In this case the two control channels need to be fully correlated and in phase opposition therefore the two method listed above are unusable.
3.2. Building the Reference Matrix

Figure 3.5 – A six-Dofs actuatur to replicate pure roll motion

Several authors tackled the problem of defining the best target possible starting from known PSD levels. The following two sections (ch. 3.2.1 and ch. 3.2.2) discuss two different state-of-art target generation procedures, suggested by Musella in [2] and Smallwood in [8]. As explained below, even if they tackle the case from different points of view both the methods show up some practical problems.

3.2.1 Eigenvalues Substitution

According to the property (a) in ch. 3.1, in case the phase and coherence profiles filled in the MIMO reference matrix will result in a non positive semi-definite matrix, it means that the eigendecomposition

\[ S_{yy}^{ref} = Q \Lambda Q' \] (3.4)

will return some eigenvalues that are negative in the frequency band of interest [18]. A quick solution would then be to force this matrix to be positive semi-definite substituting the negative eigenvalues with semi-positive ones [2]

\[ S_{yy \, new}^{ref} = Q \hat{\Lambda} Q' \] (3.5)

where \( \hat{\Lambda} \) is obtained from \( \Lambda \) by replacing the negative eigenvalues. This operation would however corrupt the PSD levels that the user wants to achieve. As explained by Musella, a minimal modification from the original matrix would be given by replacing with zeros the
negative eigenvalues, paying a rank loss equal to the number of original negative eigenvalues. Another option to preserve the rank of the original matrix would be replace the negative eigenvalues with their absolute value. Regardless of any rank-related consideration, the strong limitation of this procedure is that after having modified the eigenvalues, the newly defined reference matrix has diagonal terms that differs from the starting ones. The only option available would be to replace the PSD values with the original reference PSD levels and check if the matrix is still positive semi-definite. If this is not the case, the process can be repeated but the convergence to a positive semi-definite matrix is not guaranteed. In fig. 3.6 is shown an application example of the proposed procedure. In the figure the blue curve represents a target obtained by randomly selecting values of coherences and phases between the control channels. The matrix is non positive semi-definite as shown from in fig. 3.7 where an eigenvalue is negative in the whole frequency range. The green and magenta curves represent the modified matrix by substituting the negative eigenvalue with a zero and its absolute value, respectively. It is worth to notice that substituting the eigenvalue with a zero means a minimal modification to the original matrix. This can be seen in fig. 3.6 where the green levels are closer to the original target, compared with the magenta ones. However after substituting the PSD values with the required levels, only the substitution with the absolute value brings to a positive semi-definite target. This is shown in fig. 3.7 where one the dashed green curve is negative in the whole frequency bandwidth.

In conclusion the Eigenvalues Substitution procedure, replacing the possible negative eigenvalues, will not guarantee that the resulting PSDs will match the given levels. Substituting the obtained PSD values with the original ones instead, will not guarantee that the resulting reference matrix will be positive semi-definite and thus realisable for the control test.

3.2.2 Smallwood’s Extreme Drives Method

This section points out another target generation procedure, suggested by Smallwood in [8]. As already explained in the previous chapter
3.2. Building the Reference Matrix

Figure 3.6 – Control SDM using eigenvalues substitution procedure. Negative definite reference matrix (solid blue), positive semi-definite matrix obtained by substituting the negative eigenvalues with zero (dashed green) and its absolute value (dashed magenta). Positive semi-definite reference matrix obtained from the dashed magenta curves by further replacing the PSDs with the required levels (dashed red). [2]

Figure 3.7 – Eigenvalues of the matrices represented in 3.6. The dashed green curve refers to the reference matrix obtained by replacing the original negative eigenvalue with a zero and further replacing the PSDs with the required levels. [2]
(ch. 2), in the conduct of a MIMO Random VibCo test, the user specifies the target matrix and the control algorithm has to produce input time series that, driving the shakers and exciting the structure, are able to replicate the user defined reference matrix. It is clear that the power required by the exciters is strictly related to the target definition procedure. In this sense, starting from the required PSD levels, the purpose of Smallwood is to generate a reference matrix that minimizes (maximizes) the power provided to the shakers needed to reach the given target levels (Extreme Drive problem). The achievable exciter levels are often a limiting aspect for Multi-Exciter test and the drives minimization problem has been already mentioned in the standard practice for Multi-Axis vibration tests (MIL STD 810 G - MET 527 2014) [4].

There are several reasons for achieving this goal [9]:

- for fixed gRMS response levels, it guarantees that the exciters are minimally stressed, which means to work in safety with respect to the expensive test equipment;

- it allows to reach higher gRMS levels retarding the maximum limits of the hardware (e.g. DACs overload, amplifiers safety switch, etc) which means to have the possibility of getting the best out of the exciters performances.

An indicator for the overall power required by the exciters, for a Multi-Input system, can be the sum of the single independent drive powers. In algebraic terms, this quantity is actually related to the trace of the drives SDM (drives trace).

The idea of the Extreme Drive method proposed by Smallwood in [8] is to find, with fixed PSD levels, the set of coherences and phases between the control channels that minimize (maximize) the trace of the drives SDM, as shown in fig. 3.8.

![Figure 3.8 - Scheme of the Smallwood’s Extreme Drives Method](image)
3.2. Building the Reference Matrix

By considering the basic equation of Linear Time Invariant systems (eq. 2.12)

\[ S_{uu} = ZS_{yy}^\text{ref}Z' \]

it possible to write the diagonal terms of the eq. (eq. 2.12) as

\[ S_{uuii} = \sum_{j=1}^{\ell} \sum_{k=1}^{\ell} Z_{ik} S_{yyjk} \bar{Z}_{ij} \quad \forall i = 1 : m \]  \hspace{1cm} (3.6)

The trace of the drives SDM is by definition the sum of the diagonal terms

\[ P = Tr(S_{uu}) = \sum_{i=1}^{m} \left( \sum_{j=1}^{\ell} \sum_{k=1}^{\ell} Z_{ik} S_{yyjk} \bar{Z}_{ij} \right) = \sum_{j=1}^{\ell} \sum_{k=1}^{\ell} S_{yyjk} \sum_{i=1}^{m} \bar{Z}_{ik} Z_{ij} \]  \hspace{1cm} (3.7)

By defining the hermitian matrix \( F = Z'Z \) (carrying information about the system behaviour) and noticing that \( S_{yy}^\text{ref} \) needs to be hermitian too, eq. 3.7 can be rewritten as

\[ P = \sum_{j=1}^{\ell} S_{yyjj} F_{jj} + 2 \sum_{j=1}^{\ell-1} \sum_{i=j+1}^{\ell} |S_{yyji}| F_{ji} \cos(\phi_{ji} - \theta_{ji}) \]  \hspace{1cm} (3.8)

and by substituting the definition of CSD amplitude (eq.3.1)

\[ P = \sum_{j=1}^{\ell} S_{yyjj} F_{jj} + 2 \sum_{j=1}^{\ell-1} \sum_{i=j+1}^{\ell} \gamma_{ji} \sqrt{S_{yyj} S_{yyi}} |F_{ji}| \cos(\phi_{ji} - \theta_{ji}) \]  \hspace{1cm} (3.9)

where \( \phi_{ji} \) and \( \theta_{ji} \) are the phase angles (in radians) of the ji-th off-diagonal terms of \( S_{yy}^\text{ref} \) and \( F \), respectively. Since the PSD terms are fixed, for a given structure, the first term on the right hand side of eq. 3.9 is always positive and fixed. The second term contains the unknown quantities \( \phi_{ji} \) and \( \gamma_{ji} \) (phases and coherences of the CSDs), and can be negative because of the cosines contained in the double sum. The theoretical minimum (maximum) trace, as pointed out in [8], is obtained when the second term is minimum (maximum), i.e. when the coherences are unitary and the cosines all equal -1 (1). This observation leads to the following conditions that guarantee the theoretical minimum (maximum) drive traces:
Chapter 3. Reference Spectral Density Matrix

(I) \( P \) is minimum \( \iff \)
\[
\begin{align*}
\gamma_{ji} &= 1 \\
\phi_{ji} &= \theta_{ji} + \pi
\end{align*}
\]

(II) \( P \) is maximum \( \iff \)
\[
\begin{align*}
\gamma_{ji} &= 1 \\
\phi_{ji} &= \theta_{ji}
\end{align*}
\]

All the other possible combinations of coherences and phases will return drive traces that fall in the range between the minimum and the maximum value.

Even if the Smallwood’s Extreme Drive Method is an interesting and attractive target generation procedure, unfortunately it cannot be used to run an actual test because it does not guarantee that the resulting reference matrix is positive semi-definite, since this hypothesis is never taken into account in the two conditions pointed above.

As explained by Musella in [2] the method cannot be used to fill in a general \( \ell \times \ell \) matrix. To drive the minimum trace condition (I) in fact, all the second terms in the double summation of eq. 3.9 are forced to assume the biggest possible negative values (unitary coherences and cosines equal to -1) and eventually overcome in module the first term in the right hand side (fixed and defined by the reference PSDs and the system FRFs). This will force the resulting drives SDM to be

![Figure 3.9](image-url)

**Figure 3.9** – Theoretical drive trace for a 4x4 system using the Smallwood’s Extreme Drive Method [2]
non positive semi-definite and this property will be inherited by the reference SDM, according to the condition (f) in ch. 3.1. This limitation could be better shown by looking at the theoretical drive traces in fig. 3.9, obtained for a 4x4 system. In this plot, in logarithmic scale, the minimum drive trace assumes negative values in the bands [200-315] Hz and [410-1024] Hz. According to the condition (d) in ch. 3.1 this is sufficient to state that some eigenvalues are negative that the reference matrix is not positive semi-definite.

3.3 Phase Pivoting Method

As explained in the previous chapter, even if several authors tackled the problem to define a realisable full MIMO Random Reference matrix, a clear gap in the procedures about a generally accepted and robust method still exists.

Starting from the challenges and the limitations of previous approaches and techniques based on linear algebra, this chapter aims to assess a target definition procedure that first guarantee the target to be mathematically realizable (positive semi-definite matrix) while keeping the test specifications and then find a solution to the Extreme Drives problem.

The starting point of the method are the given PSD levels that are assumed to be known terms (standard profiles). The CSD terms are unknown and need to be defined by choosing a set of meaningful coherences and phases, in the sense that the resulting full SDM is positive semi-definite. Since a standardization of the CSD terms is not meaningful (nor possible), the idea pursued by this work is to provide a method rather than a set of values to complete the target.

Crucial for the definition of this innovative method, has been Peeters’s work [1]. Starting from the meaning of the Sylvester’s Criterion of a positive semi-definite matrix (property (c) in ch. 3.1), Peeters gives a rule to define meaningful coherence choices to get a realisable target in case of three control channels. In this case, $S_{yy}^{ref} \in \mathbb{C}^{3 \times 3}$, the Sylvester Criterion is fully fulfilled if:

(i) the coherence between two control channels is between 0 and 1;
(ii) the determinant of the full reference spectral matrix is semi-positive.

Stated that (i) is the only physical option (eq. 3.3 in ch. 3.2), to get a meaningful reference matrix is sufficient (and necessary) that

\[ \det(S_{yy}^{\text{ref}}) \geq 0 \]  \hspace{1cm} (3.10)

Referring to the typical structure of a 3x3 reference matrix (see fig. 3.2) the condition 3.10 can be written as

\[
PSD_1(PSD_2PSD_3 - CSD_{23}CSD_{23}^*) - \ldots \\
CSD_{12}(CSD_{12}PSD_3 - CSD_{13}CSD_{23}^*) + \ldots \\
CSD_{13}^*(CSD_{12}CSD_{23} - PSD_2CSD_{13}) \geq 0 \]  \hspace{1cm} (3.11)

where 1,2 and 3 are the three control channels.

Substituting the definition of CSDs as a function of PSDs, coherences and phases (eq. 3.1) and dividing the inequality by the three PSDs, the condition to get a meaningful target matrix can be rewritten as

\[
1 - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2 + 2 \cos(\phi_{12} - \phi_{13} + \phi_{23})\sqrt{\gamma_{12}^2 \gamma_{13}^2 \gamma_{23}^2} \geq 0 \]  \hspace{1cm} (3.12)

For the special case of the phase between all the control channels being zero, it is possible to fix two of the three coherences and retrieve the allowed values for the last one. For instance, fixing \( \gamma_{12} \) and \( \gamma_{13} \), eq. 3.12 allows all the \( \gamma_{23} \) values between the boundaries

\[
\gamma_{23,\text{min}} = \sqrt{\gamma_{12}^2 \gamma_{13}^2} - \sqrt{(1 - \gamma_{12}^2)(1 - \gamma_{13}^2)} \\
\gamma_{23,\text{max}} = \sqrt{\gamma_{12}^2 \gamma_{13}^2} + \sqrt{(1 - \gamma_{12}^2)(1 - \gamma_{13}^2)} \]  \hspace{1cm} (3.13a/b)

The lower and upper limits for the third coherence (\( \gamma_{23,\text{min}}, \gamma_{23,\text{max}} \) respectively) as a function of the two first coherences (\( \gamma_{12}, \gamma_{13} \)) are graphically represented in fig.3.10.

Unfortunately Peeters in [1] takes in to account only the special case of the phase between all the control channels being zero and thus the method is not general. The Phase Pivoting Method extends the condition 3.13 to more general set of phase by noticing that the two limits 3.13a and 3.13b are valid for all the phases combinations that nullify
3.3. Phase Pivoting Method

Figure 3.10 – Upper (left) and lower (right) limit for the third coherence [1]

the cosine argument in eq. 3.12. A general signal processing consideration can drive a meaningful choice of phases that will accomplish this requirement. In the generic CSD term is contained, in fact, the phase difference between pairs of control channels recordings spectra. These phase profiles cannot be independently set in case all the pairs are fully coherent. In fig. 3.11, three control channels (1, 2 and 3) are taken as example: setting the phases $\phi_{12}$ and $\phi_{13}$ means to set a relative constraint in the phase information carried by the recorded signals, i.e. a phase shift of $\phi_{12}$ between control 1 and 2 and a phase shift of $\phi_{13}$ between control 1 and 3. Thus the phase shift between the control channels 2 and 3 is unequivocally defined as the difference between $\phi_{13}$ and $\phi_{12}$. Following this principle, in case of fully coherent control pairs, the user can independently set the $(\ell - 1)$ elements of the first row (but the same could be done by picking other elements) and automatically retrieve the remaining phases by using the first row as a phase pivot

$$\phi_{i,j} = \phi_{1,j} - \phi_{1,i}$$  \hspace{1cm} (3.14)

To notice that, by following the phase pivoting principle, eq.3.12 returns always the conditions 3.13 and a positive semi-definite matrix could then always easily be obtained by choosing an appropriate coherences set (fig. 3.10).

In conclusion, to obtain a positive semi-definite reference matrix for the three control channels case is sufficient to set two phases and two coherences and then retrieve the third phase with the relation 3.14 and the third coherence in the interval defined by eq. 3.13.
An extension to the general $\ell \times \ell$ reference matrix is not straightforward because on top of the condition 3.12, $(\ell - 3)$ additional conditions need to be included to get allowable coherence boundaries. These will be given, according to the Sylvester’s Criterion, by the $(\ell - 3)$ remaining determinants. Even if a mathematical proof is still pending, in case all the control channels are fully coherent, the Phase Pivoting Method has been proven experimentally driving a six-axis test rig in different test configuration (see ch. 4) to always lead to a mathematically realisable reference matrix. This can be intuitively explained by noticing that, on top of any mathematical demonstration, there is a physical consideration driving the choice of the phases between control channels.

![Figure 3.11](image)

**Figure 3.11** – Phase Pivoting principle; Given the phases between two pairs of control channels (e.g. $\phi_{12}$ and $\phi_{13}$), the phase between the remaining pair ($\phi_{23}$) is automatically defined.

### 3.3.1 Modified Extreme Drives Method

As explained in ch. 3.2.2, the achievable exciters power levels are often a limiting aspect in the conduct of a multi-axis control test and a target generation procedure that takes in to account this problem, is a challenge both attractive and useful. The drives minimization problem has been already tackled by Smallwood in [8] but unfortunately, as explained by Musella in [2] and also mentioned in the standard practice for Multi-Exciter vibration tests [5], the method proposed by Smallwood does not guarantee a positive semi-definite reference matrix in case of more then two control channels. This translates in the practical issue of making the method not suited for testing purposes because, as already discussed in ch.2.3, the nowadays MIMO
3.3. Phase Pivoting Method

Random Control technologies make use of the Cholesky Decomposition to generate a set of multiple drives with the desired cross-correlations.

It is understandable thus, that the first aspect to be taken into account in the target definition process is its mathematical realizability. In case of fully coherent responses, it is possible to obtain a positive semi-definite matrix by randomly selecting \((\ell - 1)\) phases and derive the remaining \(\left(\frac{\ell(\ell-3)}{2} + 1\right)\) terms according to the Phase Pivoting method. This simple idea can be used to generalize the Smallwood’s Extreme Drive method.

The idea of the Extreme Drives Method proposed by Smallwood is to find, with fixed PSD levels, the set of coherences and phases between the control channels that minimize (maximize) the trace of the drives SDM (fig. 3.8). The main issue of the method derives by the fact that, in case of fully coherent responses, the phase profiles cannot be independently set but need to follow the Phase Pivoting principle. This means that out of the \(\left(\frac{\ell(\ell+1)}{2} - \ell\right)\) relative phases to be set, just \((\ell - 1)\) can be independently picked, the remaining \(\left(\frac{\ell(\ell-3)}{2} + 1\right)\) terms need to be pivoted from the ones set.

In order to solve this issue and to obtain the realizable target that still requires the minimum (maximum) drives, the idea is to choose ad-hoc the pivot phases. For a better understanding, the formula to obtain the drives SDM trace, starting from the Reference SDM, is again shown below (eq. 3.9 in ch. 3.2.2)

\[
P = \sum_{j=1}^{\ell} S_{yyjj} F_{jj} + 2 \sum_{j=1}^{\ell-1} \sum_{i=j+1}^{\ell} \sqrt{S_{yyj} S_{yyi}} |F_{ji}| \cos(\phi_{ji} - \theta_{ji})
\]

By looking at eq. 3.9, the term that contributes to minimize (maximize) the drives trace is the double sum in the right hand side. This term is a summation of contributions coming from the \(ij\)-th cross terms with different magnitudes \(2\sqrt{S_{yyj} S_{yyi}} |F_{ji}| \cos(\phi_{ji} - \theta_{ji})\). The best solution would then be to use as phase pivots the \((\ell - 1)\) terms with the biggest amplitude (i.e. assign their phases according to the Smallwood’s method, conditions (I)(II) in ch. 3.2.2), that thus will decrease (increase) by the maximum amount the fixed positive term (first term on the right hand of eq. 3.9). The phases of the remaining terms will be pivoted from the remaining ones.

The Modified Extreme Drive method to obtain the minimum trace
of the drives SDM, is schematically represented in fig. 3.12. In this practical example, three control channels are used and a 3x3 Reference SDM needs to be defined. Applying condition (I) in ch. 3.2.2 and computing the magnitudes of the three cross-terms (eq. 3.9), the resulting term with the smallest amplitude (i.e. the one that gives the minor contribution to minimize the trace) is $C_{SD_{13}}$. For this term the Phase Pivoting principle is applied and thus a positive semi-definite matrix, that generates the minimum possible drives trace, is obtained. Fig. 3.13 instead, schematically shows the procedure for the maximum trace of the drives matrix; the method is completely the same, but in this case condition (II) in ch. 3.2.2 is applied.

Both the innovative target generation procedures, the Phase Pivoting method and the Modified Extreme Drives method, have been experimentally tested using a 6-DOFs hydraulic shaking system and the results are shown in the following chapter.
3.3. Phase Pivoting Method

**Figure 3.12** – Modified Extreme Drives procedure to obtain the minimum trace of the drives SDM

**Figure 3.13** – Modified Extreme Drives procedure to obtain the maximum trace of the drives SDM
Chapter 4

Test Cases

In the previous chapter, two innovative target generation procedures have been theoretically demonstrated. Both the methods have been implemented into an algorithm able to automatically generate realisable full Reference SDMs, starting from some user defined choices. The algorithm has been tasted with a series of experiments and the results are discussed in the following.

This work is part of the paper titled "Tackling the target matrix definition in MIMO Random Vibration Control testing" [9], submitted and approved for the 30th Aerospace Testing Seminar and some of the testing results are shown in the paper, too.

4.1 Test Setup

The MIMO Random Vibco tests have been carried out on a 6-DOFs hydraulic shaking system, the so-called CUBETM installed in the Noise and Vibration Laboratory at the KU Leuven University (fig. 4.1).

![Figure 4.1 – The CUBE: a 6-DOFs hydraulic shaking system installed at KU Leuven University](image)
The CUBE shaker table is mounted on a reaction mass of about 32 ton (fig. 4.2), suspended by 6 air springs with automatic mechanical leveling system, highly increasing the systems isolation capacity [19]. Within the movable CUBE are six servo-hydraulic actuators with hydrostatic bearings connecting the actuators to the structure. The six actuators are arranged in pairs, one pair on each of X, Y and Z-axes, allowing a full 6-DOF motion in modal control (fig. 4.3):

- 3 translations $\Rightarrow \begin{cases} T_x \ (\text{Drive: Output 1}) \\ T_y \ (\text{Drive: Output 2}) \\ T_z \ (\text{Drive: Output 3}) \end{cases}$

- 3 rotations $\Rightarrow \begin{cases} Roll \ (\text{Drive: Output 4}) \\ Pitch \ (\text{Drive: Output 5}) \\ Yaw \ (\text{Drive: Output 6}) \end{cases}$

Combined with high frequency response servo-valves, this configuration has demonstrated controllable excitation to 250 Hz in all axes [19]. Some other specification are given in tab. 4.1.

<table>
<thead>
<tr>
<th>DOF</th>
<th>Stroke (mm)</th>
<th>Acceleration (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_x$</td>
<td>50.8</td>
<td>6.8</td>
</tr>
<tr>
<td>$T_y$</td>
<td>50.8</td>
<td>4.4</td>
</tr>
<tr>
<td>$T_z$</td>
<td>101.6</td>
<td>5.3</td>
</tr>
<tr>
<td>Roll</td>
<td>5°</td>
<td>n/a</td>
</tr>
<tr>
<td>Pitch</td>
<td>4.5°</td>
<td>n/a</td>
</tr>
<tr>
<td>Yaw</td>
<td>6°</td>
<td>n/a</td>
</tr>
</tbody>
</table>

Table 4.1 – Performance specification CUBE [19]

The tests have been performed using the LMS SCADAS Mobile as data acquisition hardware and MIMO Random, the Siemens LMS Test.Lab vibration control software (fig. 4.4).

4.1.1 Sensors Configuration

During the test campaign, three different sensor configurations have been used (fig. 4.5):
• **Configuration 1**
  Three accelerometers are placed at the corners of the shaker’s top surface and three drives are used to recreate the motion (tab. 4.2, fig. 4.5).

<table>
<thead>
<tr>
<th>DRIVES</th>
<th>ctrl CHANNELS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_x$</td>
<td>CH 1:+Z</td>
</tr>
<tr>
<td>Roll</td>
<td>CH 2:+Z</td>
</tr>
<tr>
<td>Pitch</td>
<td>CH 3:+Z</td>
</tr>
</tbody>
</table>

*Table 4.2 – Inputs Outputs Configuration 1*

• **Configuration 2**
  Two accelerometers are placed at the shaker’s right side surface, in correspondence of the actuators’ heads, and one at the center of the top surface. Only two drives are involved (tab. 4.3, fig. 4.5).

<table>
<thead>
<tr>
<th>DRIVES</th>
<th>ctrl CHANNELS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_z$</td>
<td>CH 9:+Y</td>
</tr>
<tr>
<td>CH 10:+Y</td>
<td></td>
</tr>
<tr>
<td>Roll</td>
<td>CH 0:+Y</td>
</tr>
</tbody>
</table>

*Table 4.3 – Inputs Outputs Configuration 2*

• **Configuration 3**
  Six accelerometers are placed in correspondence of the six actuators’ heads and all the drives are involved (tab. 4.4, fig. 4.5).

<table>
<thead>
<tr>
<th>DRIVES</th>
<th>ctrl CHANNELS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_x$</td>
<td>CH 5:+Z</td>
</tr>
<tr>
<td>$T_y$</td>
<td>CH 6:+Z</td>
</tr>
<tr>
<td>$T_z$</td>
<td>CH 7:+X</td>
</tr>
<tr>
<td>Roll</td>
<td>CH 8:+X</td>
</tr>
<tr>
<td>Pitch</td>
<td>CH 9:+Y</td>
</tr>
<tr>
<td>Yaw</td>
<td>CH 10:+Y</td>
</tr>
</tbody>
</table>

*Table 4.4 – Inputs Outputs Configuration 3*
Chapter 4. Test Cases

Figure 4.2 – The CUBE reaction mass [19]

Figure 4.3 – Six servo-hydraulic actuators allow the 6-DOF motion [19]

Figure 4.4 – MIMO CUBE Control Setup

Figure 4.5 – Configuration 1 (left); Configuration 2 (middle); Configuration 3 (right).
4.2 Pre-Test Phase

In the pre-test phase, the System Identification and the System Verification are carried out regardless of the controlled test to perform.

4.2.1 System Identification

The first step for a MIMO vibration control test is the estimation of the system’s FRFs (ch. 2.3). Loosely speaking, this step is necessary to understand how many g’s the accelerometers controlled recordings (ctrl CHANNELS) will pick up per unit Volt sent by each controller’s output channels (DRIVES). The FRFs of the three different sensor configurations are shown in fig. 4.7. Notice that, each column of the FRF matrix is relative to one drive, and each row to one control channel. For the Configuration 2 for instance, three control channels and only two drives are involved and thus a rectangular system is defined (case (b) fig. 4.7).

4.2.2 System Verification

To verify the system in environmental testing applications means to predict the voltage levels, and more generally the system’s behavior, considering the target set and given the system FRFs identified in the random pre-test phase. This phase is very important for several reasons [16]: (i) it allows to determine if the target set is mathematically realizable, (ii) it allows to determine possible overload of the Data Acquisition System, due for instance to extremely high levels required (an example is shown in fig. 4.6) and (iii) to identify frequency ranges where the control performances could be critical.

Figure 4.6 – DAC overload detected on the System Verification phase
Figure 4.7 – FRFs from voltages to acceleration of the three sensors configurations
4.3 Application of the Phase Pivoting Method

To verify the Phase Pivoting method, four different tests are carried out. In all the performed tests the required levels are 0.4 gRMS white-pink noise in the band [18.75 - 150] Hz, for all the channels. These levels will be fixed for all the tests, being representative for the test requirements. The frequency resolution is 3.125 Hz. For safety reasons all the tests are performed in three sequential steps (-9 dB, -6 dB, -3 dB and full level). All the results are shown at normal ends, meaning that the tests have been safely (no abort whatsoever) run for 1 minute at the full level.

4.3.1 Test 1

The test is performed using the Configuration 1 (tab. 4.2, fig. 4.5) and the resulting FRFs are shown in fig. 4.7 (a). Three control channels (recording in z-direction) are used, therefore a realisable 3x3 Reference SDM needs to be defined. The motion that the CUBE needs to replicate is schematically represented in three different time instants in fig. 4.8, where the maximum and minimum displacement are 1 and 0 respectively.

![Figure 4.8 – Schematic representation of the motion reproduced in Test 1](image)

To control the CUBE in that way, fully coherent responses with specific phase shifts between the control channels need to be setted, e.g. 90° between CH1 and CH2, 45° between CH1 and CH3 and −45° between CH2 and CH3. Notice that the Phase Pivoting principle is perfectly respected

\[
\phi_{13} - \phi_{12} = \phi_{23} \Rightarrow 45^\circ - 90^\circ = -45^\circ
\]  

(4.1)

Setting the required PSD levels (0.4 gRMS white-pink noise) as diagonal terms and filling in the off-diagonal terms with high coherence (\(\gamma = 0.98\) for all the CSDs) and with the phase values discussed
in eq. 4.2, returns the Reference SDM shown in fig. 4.9, where exploiting the symmetry of the matrix, the lower part is used to visualize the phases of the upper off-diagonal CSDs terms. Following the Phase Pivoting method, the target is a positive semi-definite matrix (as proven by its eigenvalues in fig. 4.10) and it is thus possible to run the control test. Fig. 4.11 shows the SDM of the control channels for the normal end test, where the orange dash line is the alarm limit, fixed at $\pm 3dB$ from the reference, and the red line is the abort limit, fixed at $\pm 6dB$.

**Figure 4.9** – Reference SDM Test 1: $\gamma_{12} = \gamma_{13} = \gamma_{23} = 0.98$ and $\phi_{12} = 90^\circ$, $\phi_{13} = 45^\circ$, $\phi_{23} = -45^\circ$.

**Figure 4.10** – Eigenvalues of the Reference SDM in Test 1
4.3. Application of the Phase Pivoting Method

4.3.2 Test 2

Through this test you want to point out the advantages in terms of control results on using the Phase Pivoting method. Adopting the Configuration 1, the two Reference SDMs represented in fig. 4.12 want to be compared. The case (a) in fig. 4.12 is the same reference matrix used in Test 1 except that the coherence values are setted at 0.5 for all the CSD terms. The case (b) in fig. 4.12 instead, is a target matrix that does not follow the Phase Pivoting principle. Notice that, in order to compare the control results of the two targets, a coherence lowering is necessary because of with fully coherent responses the case (b) returns a negative matrix and the test cannot be run. Fig. 4.13 shows the results for the two normal end control tests. As you can notice, the control results are significantly better for the case (a). The case (b) in fact, shows up several spectral line out of the abort limit, symptom of a not entirely correct target generation procedure. Special attention should be given on the phases behaviour: the control system is not able to replicate the reference phases due to fact that the required phase shifts are in conflict with each other and thus have not a real physical meaning.
Figure 4.12 – Reference SDMs Test 2: (a) follows the Phase Pivoting principle, (b) does not follow the Phase Pivoting principle.

(a) SDM of the control channels Test 2

(b) SDM of the control channels Test 2

Figure 4.13 – SDMs of the control channels Test 2: (a) follows the Phase Pivoting principle, (b) does not follow the Phase Pivoting principle.
4.3. Application of the Phase Pivoting Method

4.3.3 Test 3

Test 3 is carried out to underline a possible problem related to the misuse of the Phase Pivoting method.

The test is performed using the Configuration 2 (tab. 4.3, fig. 4.5) and the resulting FRFs are shown in fig. 4.7 (b). To run the control test, the 3x3 Reference SDM represented in fig. 4.14 is defined following the Phase Pivoting principle

\[
\phi_{13} - \phi_{12} = \phi_{23} \Rightarrow 180^\circ - 0^\circ = 180^\circ \quad (4.2)
\]

![Figure 4.14 – Mathematically realisable Reference SDM Test 3](image)

The resulting SDM of the control channels is shown in fig. 4.16. As you can notice, the control system is not able to replicate the required Reference therefore the test has been stopped before the normal end because of to many spectral lines are over the abort limits (control abort).

These bad control results are due to the fact that the reference matrix is just mathematically realisable but not physically realisable for the control system. The target in fact, even if is a positive semi-definite matrix because it follows the Phase Pivoting principle, it does not take into account the physical constraints related to the particular sensors configuration setup used for test. Set the channel CH0 in phase opposition with respect to the other two channels (CH9, CH10) it is not allowed because the top surface and the side surface of the CUBE are physically constrained (fig. 4.15).

![Figure 4.15 – Physical constraint related to channel setup](image)
In order to perform a true MDOF laboratory motion replication, understanding the crucial role of the system dynamics related to the spatial locations of reference transducers, another Reference SDM is defined (fig. 4.17). This target, in addition to follow the Phase Pivoting principle, it also has a clear physical meaning because it generates the roll motion of the CUBE without violating any system constraints. The excellent results for the normal end control test using the new Reference is shown in fig. 4.18, where no spectral line is out of the alarm limit fixed at $\pm 3dB$ from the reference for all terms.

<table>
<thead>
<tr>
<th>$\text{PSD}_{c9}$</th>
<th>0.98</th>
<th>0.98</th>
</tr>
</thead>
<tbody>
<tr>
<td>$180^\circ$</td>
<td>$\text{PSD}_{c10}$</td>
<td>0.98</td>
</tr>
<tr>
<td>$180^\circ$</td>
<td>$0^\circ$</td>
<td>$\text{PSD}_{c0}$</td>
</tr>
</tbody>
</table>

Figure 4.17 – Physically realisable Reference SDM Test 3
4.3. Application of the Phase Pivoting Method

Figure 4.18 – Physically realisable SDM of the control channels Test 3

4.3.4 Test 4

In ch. 3.3 the Phase Pivoting method is mathematically demonstrated for the only case of 3x3 Reference SDMs because, an extension to the general $\ell$-dimensional case is not straightforward due to the $(\ell - 3)$ extra conditions that need to be included in the demonstration. Test 4 is thus carried out to prove the validity of the method also in the case where more than 3 control channels are used.

The actual test is performed considering the Configuration 3 illustrated in fig. 4.5, where 6 channels are simultaneously controlled (the most representative FRFs coming from the System Identification phase are shown in fig. 4.7 (c)).

From the implementation point of view, the easier way to obtain a realisable full target following the Phase Pivoting principle, can be by setting the elements of the first row and retrieve the remaining phases by using the first row as a phase pivot (eq. 3.14). This technique is used to generate the fully coherent Reference SDM represented in fig. 4.19, where the first row of the matrix is defined ad-hoc in order to have specific phase shifts between the control channel pairs placed at the three different CUBE’s surfaces (green elements of the matrix). This positive semi-definite matrix can be used to run
the test, and the resulting SDM of the control channels is shown in fig. 4.21. As you can notice in fig. 4.20, the normal end test is completed with just two spectral lines out of the alarm limit and no spectral line out of the abort limit (fixed at $\pm 3\, \text{dB}$ and $\pm 6\, \text{dB}$ respectively). The hight quality of the control results is even more significant considering that 6 channels are involved and thus 36 elements are simultaneously controlled.

![Figure 4.19 – 6x6 Reference SDM Test 4](image1)

![Figure 4.20 – Normal end Test 4: two spectral lines out of the abort, no spectral line out of the alarm](image2)
4.3. Application of the Phase Pivoting Method

Figure 4.21 – 6x6 SDM of the control channels Test 4
4.4 Application of the Modified Extreme Drives Method

The Modified Extreme Drives method, explained and demonstrated in ch. 3.3.1, is experimentally proven through the following two control tests. In both the tests and for all the control channels, the required levels are 0.4 gRMS white-pink noise in the band [18.75 - 150] Hz, with a frequency resolution of 3.125 Hz. For safety reasons all the tests are performed in three sequential steps (-9 dB, -6 dB, -3 dB and full level) and the results are shown at normal ends (1 minute at full level).

4.4.1 Test 5

Test 5 is performed using the Configuration 1 (tab. 4.2, fig. 4.5) and the resulting FRFs are shown in fig. 4.7 (a). Three control channels are used, therefore a realisable 3x3 Reference SDM that minimizes (maximizes) the trace of the drives SDM, needs to be defined. Before the actual test a system verification can be performed in order to check, with the information coming from the System Identification, the predicted levels for the responses and the drives. Fig. 4.22 clearly illustrates how, setting the phases with the proposed Minimum Drives method, the exciters’ \( V_{RMS} \) drastically decreases. An uniform reduction of the single drives is not guaranteed by the method that actually optimizes the sum of the required power, as pointed out in fig. 4.22, where the sum of the RMS gives an indication of the actual reduced quantity. On the other hand setting the phases with the Maximum Drives method returns much higher predicted drives levels. The figure also illustrates that uncorrelated responses (i.e. low coherences) require more energy than fully correlated responses with an ad-hoc choice of phases.

In fig. 4.25 the SDMs of the control channels are represented for the normal end test performed by setting the reference matrix with the Minimum Drives method (a) and with the Maximum Drives method (b). The focus, more then on the achieved control results, needs however to go on the required drives power. The drives traces calculated for these two tests (minimum and maximum) are shown in fig. 4.23, where the single drives’ PSDs are presented too. There is clearly a big
4.4. Application of the Modified Extreme Drives Method

gap (up to two order of magnitude) in terms of achieved drives traces obtained using the Modified Extreme Drives method (minimum and maximum). This gap reflects the overall behaviour of reducing the drives PSDs predicted in the system verification phase (fig. 4.22), and it is a concrete evidence of the method’s validity. As a further proof, additional simulations results are presented in fig. 4.24. It is worth to notice that any other phase choice (most probably) and coherence (certainly) will generally return drives traces that fall in between the Extreme Drives ones.

Figure 4.22 – Predicted drives $V_{RMS}$ from a system’s verification Test 5

Figure 4.23 – Drives PSDs and drives traces using the Modified Extreme Drives method Test 5
Figure 4.24 – Simulated drives traces Test 5

4.4.2 Test 6

Test 6 is performed considering the Configuration 3 illustrated in fig. 4.5, where 6 channels are simultaneously controlled (the most representative FRFs coming from the System Identification phase are shown in fig. 4.7 (c)).

The actual test is thus carried out to prove the validity of the Modified Extreme Drives method also in the case where more than 3 control channels are used. In fact, even if in the case of three control channels the algorithm’s implementation has been straightforward and guarantees the exact solution to the problem, a generalization of the method for \( \ell \)-control channels (\( \ell > 3 \)) is quite challenging. An approximate tested solution is to use, as a phase pivot, the phases of the first row coming from the Smallwood’s reference matrix, and apply the Phase Pivoting principle to complete the remaining phases, as shown in fig. 4.26.
4.4. Application of the Modified Extreme Drives Method

Figure 4.25 – SDMs of the control channels using the Modified Extreme Drives method Test 5

(a) SDM of the control channels with Minimum Drives method Test 5

(b) SDM of the control channels with Minimum Drives method Test 5
Fig. 4.26 – Reference SDM for the minimum drives requirement using as phase pivot the first row of the matrix Test 6

Fig. 4.27 clearly points out that even this approximate solution gives a clear advantage in terms of achievable test levels. In these demo case the test PSDs are scaled up to get a 3 gRMS level for each of the control channels. These levels are not achievable in case the reference SDM is filled in to get fully uncorrelated responses, because the voltage to be sent out from the acquisition system’s DAC that controls the pitch is out of the safety threshold (DAC overload). The solution proposed is thus to set the phases between the control channels with the Minimum Drives method. In this case, the voltage predicted for the pitch DAC is approximately two times lower and 1 V below the DAC overload threshold. The figure also shows that the situation gets worse by setting the phases with the Maximum Drives method.

Fig. 4.27 – Predicted drives $V_{RMS}$ from a system’s verification Test 6

Having shown the capability of the approach, the actual tests is run and the control results for the minimum trace requirement are shown in fig. 4.29. As you can notice, minimize the required drives power is
translated in a loss of quality in the control of the CSD terms. However the overall control results remain more than acceptable due to an excellent control of the PSD terms.

The resulting minimum and maximum drives traces are shown in fig. 4.28, where the advantage in terms of power required in the overall band is substantial: the RMS value of the minimum is about a third less with respect to the maximum. Notice that, even if using the first row as a phase pivot is the easiest way to generate the target, it is just an approximation of the method and it does not guarantee the absolute minimum and maximum trace values because possible more favourable solutions could exist.

![Drives Matrix Trace](image)

**Figure 4.28** – Drives traces using the Modified Extreme Drives method with the first row approximation Test 6
Figure 4.29 – 6x6 SDM of the control channels Test 6
Chapter 5

Conclusions

This work aims to find a general solution to one of the most challenging questions behind the multi-axis Random Vibration Control tests: the target generation procedure. Even if several author tackled the problem to define a realisable full MIMO Random Reference, a clear gap in the standard procedures about a generally accepted and robust method still exists. Before investigating possible advanced target definition scenarios in fact, there is the fundamental need to guarantee that the operation of filling in element by element the off-diagonal terms will return a matrix that has a physical meaning. This is translated in the algebraic requirement that the Reference Spectral Density Matrix must be positive semi-definite. In this sense, a more innovative target generation procedure has been proposed in this work: the Phase Pivoting method. Following the principle behind the method allows to overcome the algebraic constraint and a mathematically realisable reference matrix (positive semi-definite) is always defined. The method has been demonstrated only in the case where three control channels are used and to prove its validity also for more general cases, a series of tests has been carried out on a 6-DOFs hydraulic shaking system (CUBE\textsuperscript{TM}). The tests have confirmed the consistency of the method, and finally the Cholesky Decomposition of the Reference is no longer a problem. About the quality of the control results, it is important to underline that the Phase Pivoting method does not guarantee the physical realisability of the Reference and, in order to have the best control performances, more considerations related to the sensors configuration used to perform the test need to be included. It is only by combining the Phase Pivoting principle and a keen awareness of the system dynamics with respect to the spatial locations of the reference transducers, that allows you to carry out a
true MDOF laboratory motion replication with high control performances.

The Phase Pivoting principle has been then used to overcome the limitations related to the achievable exciter power levels performing multi-exciters control tests. With respect to this problem, starting from the Smallwood’ work, the Modified Extreme Drives method has been proposed. Setting the required PSD levels, this target generation procedure is able to return a full reference matrix with the additional attractive feature of minimize (maximize) the drives matrix trace, which means minimize (maximize) the overall power required by the shakers. The tests carried out on the CUBE\textsuperscript{TM} have pointed out clear advantages in term of drive power reduction on using the Minimum Drives method, resulting in lowering the risk of possible overloads of the Data Acquisition System. Notice that, in the case where three control channels are used, the method guarantees the exact solution of the minimum (maximum) drives trace. For the general case where more than three control channels are involved instead, the first row approximation has been adopted simplifying the algorithm implementation. Even if this approximation solution has given excellent results in term of achievable power test levels, more favourable implementation solutions could exist. These considerations leave room to further proceed in improving the Modified Extreme Drives method reaching thus the best solution possible.
Bibliography


