ON COLEMAN AUTOMORPHISMS OF FINITE GROUPS AND THEIR MINIMAL NORMAL SUBGROUPS

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Definition
Let $G$ be a finite group and $\sigma \in \text{Aut}(G)$. If for any prime $p$ dividing the order of $G$ and any Sylow $p$-subgroup $P$ of $G$, there exists a $g \in G$ such that $\sigma|_P = \text{conj}(g)|_P$, then $\sigma$ is said to be a Coleman automorphism.
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Denote $\text{Aut}_{\text{col}}(G)$ for the set of Coleman automorphisms, $\text{Inn}(G)$ for the set of inner automorphisms and set

$$\text{Out}_{\text{col}}(G) = \frac{\text{Aut}_{\text{col}}(G)}{\text{Inn}(G)}.$$
Theorem (Hertweck and Kimmerle)

Let $G$ be a finite group. The prime divisors of $|\text{Aut}_{\text{col}}(G)|$ are also prime divisors of $|G|$. 
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Lemma

Let $N$ be a normal subgroup of a finite group $G$. If $\sigma \in \text{Aut}_{\text{col}}(G)$, then $\sigma|_N \in \text{Aut}(N)$. 
Let $G$ be a group and $R$ a ring (denote $U(RG)$ for the units of $RG$), then we clearly have that

$$GC_{U(RG)}(G) \subseteq N_{U(RG)}(G)$$
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Is this an equality?

$$N_{U(RG)}(G) = GC_{U(RG)}(G)$$

Of special interest: $R = \mathbb{Z}$
If $u \in N_{U(RG)}(G)$, then $u$ induces an automorphism of $G$:

$$\varphi_u : G \rightarrow G$$

$$g \mapsto u^{-1}gu$$
If $u \in N_{U(R_G)}(G)$, then $u$ induces an automorphism of $G$:

$$\varphi_u : G \to G$$

$$g \mapsto u^{-1}gu$$

Denote $\text{Aut}_U(G; R)$ for the group of these automorphisms ($\text{Aut}_U(G)$ if $R = \mathbb{Z}$) and $\text{Out}_U(G; R) = \text{Aut}_U(G; R)/\text{Inn}(G)$
Theorem (Jackowski and Marciniak)

Let $G$ be a finite group, $R$ a commutative ring. TFAE

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Theorem

Let $G$ be a finite group. Then,

$$\text{Aut}_U(G) \subseteq \text{Aut}_{col}(G)$$
Theorem (Hertweck)

There exists a finite metabelian group $G$ of order $2^{25}97^2$ with

$$\text{Aut}_{U(RG)}(G) \neq \text{Inn}(G).$$
Lemma (Hertweck)

Let $p$ be a prime number and $\alpha \in \text{Aut}(G)$ of $p$-power order. Assume that there exists a normal subgroup $N$ of $G$, such that $\alpha|_N = \text{id}_N$ and $\alpha$ induces identity on $G/N$. Then $\alpha$ induces identity on $G/O_p(Z(N))$. Moreover, if $\alpha$ also fixes a Sylow $p$-subgroup of $G$ elementwise, i.e. $\alpha$ is $p$-central, then $\alpha$ is conjugation by an element of $O_p(Z(N))$. 
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Theorem (Hertweck and Kimmerle)

For any finite simple group $G$, there is a prime $p$ dividing $|G|$ such that $p$-central automorphisms of $G$ are inner automorphisms.
Theorem (A.V.A.)

Let $G$ be a normal subgroup of a finite group $K$. Let $N$ be a minimal non-trivial characteristic subgroup of $G$. If $C_K(N) \subseteq N$, then every Coleman automorphism of $K$ is inner. In particular, the normalizer problem holds for these groups.
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Proof.

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SELF-CENTRALITY

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Proof.

1. $N$ is characteristically simple
2. Restrict to simple groups
3. Subdivide Abelian vs. Non-abelian
4. Hertweck’s result
Corollary

Let $G = P \rtimes H$ be a semidirect product of a finite $p$-group $P$ and a finite group $H$. If $C_G(P) \subseteq P$, then $G$ has no non-inner Coleman automorphisms.
The wreath product $G \wr H$ of $G, H$ is defined as $\prod_{h \in H} G \rtimes H$, where $H$ acts on the indices.
COROLLARY 2

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Corollary

Let $S$ be a finite simple group, $I$ a finite set of indices, and $H$ any finite group. If $G = \prod_{i \in I} S \rtimes H$ is a group such that $C_G(\prod_{i \in I} S) \subseteq Z(\prod_{i \in I} S)$. Then $G$ has no non-inner Coleman automorphism. In particular, the normalizer problem holds for $G$. Then, in particular, the wreath product $S \wr H$ has no non-inner Coleman automorphisms. Moreover, in both cases the normalizer problem holds for $G$. 
Questions (Hertweck and Kimmerle)

1. Is $\text{Out}_{\text{col}}(G)$ a $p'$-group if $G$ does not have $C_p$ as a chief factor?
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Partial Answer (Hertweck, Kimmerle)

Besides giving several conditions, all three statements hold if $G$ is assumed to be a $p$-constrained group.
Partial Answers (Van Antwerpen)

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2. True, if $O_p(G) = O_{p'}(G) = 1$, where $p$ is an odd prime and the order of every direct component of $E(G)$ is divisible by $p$.
3. True, if the unique minimal non-trivial normal subgroup is non-abelian. True, if question 2 has a positive answer.
Inkling of Idea

In case $G$ has a unique minimal non-trivial normal subgroup $N$, which we may assume to be abelian, we may be able to use the classification of the simple groups and the list of all Schur multipliers to give a technical proof.
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Related Question

Gross conjectured that for a finite group G with $O_{p'}(G) = 1$ for some odd prime p, p-central automorphisms of p-power order are inner. Hertweck and Kimmerle believed this is possible using the classification of Schur multipliers.
Stretch of Idea

M. Murai connected Coleman automorphisms of finite groups to the theory of blocks in representation theory, showing several slight generalizations of known theorems. Looking into his methods may prove useful.
REFERENCES

4. E.C. Dade. Locally trivial outer automorphisms of finite groups.
5. A. Van Antwerpen. Coleman automorphisms of finite groups and their minimal normal subgroups. Arxiv preprint. Accepted in JPPA