CHARACTERIZING THE GROUP OF COLEMAN AUTOMORPHISMS

Arne Van Antwerpen
August 21, 2017
Definition
Let $G$ be a finite group and $\sigma \in \text{Aut}(G)$. If for any prime $p$ dividing the order of $G$ and any Sylow $p$-subgroup $P$ of $G$, there exists a $g \in G$ such that $\sigma|_P = \text{conj}(g)|_P$, then $\sigma$ is said to be a Coleman automorphism.
Definition

Let $G$ be a finite group and $\sigma \in \text{Aut}(G)$. If for any prime $p$ dividing the order of $G$ and any Sylow $p$-subgroup $P$ of $G$, there exists a $g \in G$ such that $\sigma|_P = \text{conj}(g)|_P$, then $\sigma$ is said to be a Coleman automorphism.

Denote $\text{Aut}_{col}(G)$ for the set of Coleman automorphisms, $\text{Inn}(G)$ for the set of inner automorphisms and set

$$\text{Out}_{col}(G) = \frac{\text{Aut}_{col}(G)}{\text{Inn}(G)}.$$
Theorem (Hertweck and Kimmerle)

Let $G$ be a finite group. The prime divisors of $|\text{Aut}_{\text{col}}(G)|$ are also prime divisors of $|G|$. 
Theorem (Hertweck and Kimmerle)

Let $G$ be a finite group. The prime divisors of $|\text{Aut}_{\text{col}}(G)|$ are also prime divisors of $|G|$.

Lemma

Let $N$ be a normal subgroup of a finite group $G$. If $\sigma \in \text{Aut}_{\text{col}}(G)$, then $\sigma|_N \in \text{Aut}(N)$. 
Let $G$ be a group and $R$ a ring (denote $U(RG)$ for the units of $RG$), then we clearly have that

$$GC_{U(RG)}(G) \subseteq N_{U(RG)}(G)$$
Let $G$ be a group and $R$ a ring (denote $U(RG)$ for the units of $RG$), then we clearly have that

$$GC_{U(RG)}(G) \subseteq N_{U(RG)}(G)$$

Is this an equality?

$$N_{U(RG)}(G) = GC_{U(RG)}(G)$$

Of special interest: $R = \mathbb{Z}$
If \( u \in N_{U(RG)}(G) \), then \( u \) induces an automorphism of \( G \):

\[
\varphi_u : G \rightarrow G
\]

\[
g \mapsto u^{-1}gu
\]
If $u \in N_{U(RG)}(G)$, then $u$ induces an automorphism of $G$:

$$\varphi_u : G \rightarrow G$$

$$g \mapsto u^{-1}g u$$

Denote $\text{Aut}_U(G; R)$ for the group of these automorphisms ($\text{Aut}_U(G)$ if $R = \mathbb{Z}$) and $\text{Out}_U(G; R) = \text{Aut}_U(G; R)/\text{Inn}(G)$
Theorem (Jackowski and Marciniak)

$G$ a finite group, $R$ a commutative ring. TFAE

1. $N_{U(RG)}(G) = GC_{U(RG)}(G)$
2. $\text{Aut}_U(G; R) = \text{Inn}(G)$
Theorem (Jackowski and Marciniak)

Let $G$ be a finite group, $R$ a commutative ring. TFAE

1. $N_{U(RG)}(G) = GC_{U(RG)}(G)$
2. $\text{Aut}_U(G; R) = \text{Inn}(G)$

Theorem

Let $G$ be a finite group. Then,

$$\text{Aut}_U(G) \subseteq \text{Aut}_{col}(G)$$
Theorem (Hertweck)

There exists a finite metabelian group $G$ of order $2^597^2$ with

$$\text{Aut}_{U(RG)}(G) \neq \text{Inn}(G).$$
GENERALIZED DIHEDRAL GROUPS
NILPOTENT-BY-CYCLIC GROUPS
Questions (Hertweck and Kimmerle)

1. Is $\text{Out}_{\text{col}}(G)$ a $p'$-group if $G$ does not have $C_p$ as a chief factor?
Questions (Hertweck and Kimmerle)

1. Is $\text{Out}_{\text{col}}(G)$ a $p'$-group if $G$ does not have $C_p$ as a chief factor?
2. Is $\text{Out}_{\text{col}}(G)$ trivial if $O_{p'}(G)$ is trivial?
Questions (Hertweck and Kimmerle)

1. Is $\text{Out}_{col}(G)$ a $p'$-group if $G$ does not have $C_p$ as a chief factor?

2. Is $\text{Out}_{col}(G)$ trivial if $O_{p'}(G)$ is trivial?

3. Is $\text{Out}_{col}(G)$ trivial if $G$ has a unique minimal non-trivial normal subgroup?

Partial Answer (Hertweck, Kimmerle)

Besides giving several conditions, all three statements hold if $G$ is assumed to be a $p$-constrained group.
Questions (Hertweck and Kimmerle)

1. Is $\text{Out}_{col}(G)$ a $p'$-group if $G$ does not have $C_p$ as a chief factor?

2. Is $\text{Out}_{col}(G)$ trivial if $O_{p'}(G)$ is trivial?

3. Is $\text{Out}_{col}(G)$ trivial if $G$ has a unique minimal non-trivial normal subgroup?

Partial Answer (Hertweck, Kimmerle)

Besides giving several conditions, all three statements hold if $G$ is assumed to be a $p$-constrained group.
Partial Answers (Van Antwerpen)

1. *No new result.*
Partial Answers (Van Antwerpen)

1. No new result.

2. True, if $O_p(G) = O_{p'}(G) = 1$, where $p$ is an odd prime and the order of every direct component of $E(G)$ is divisible by $p$. 
Partial Answers (Van Antwerpen)

1. No new result.
2. True, if $O_p(G) = O_{p'}(G) = 1$, where $p$ is an odd prime and the order of every direct component of $E(G)$ is divisible by $p$.
3. True, if the unique minimal non-trivial normal subgroup is non-abelian. True, if question 2 has a positive answer.
Inkling of Idea

In case $G$ has a unique minimal non-trivial normal subgroup $N$, which we may assume to be abelian, we may be able to use the classification of the simple groups and the list of all Schur multipliers to give a technical proof.
Inkling of Idea

In case $G$ has a unique minimal non-trivial normal subgroup $N$, which we may assume to be abelian, we may be able to use the classification of the simple groups and the list of all Schur multipliers to give a technical proof.

Related Question

Gross conjectured that for a finite group $G$ with $O_{p'}(G) = 1$ for some odd prime $p$, $p$-central automorphisms of $p$-power order are inner. Hertweck and Kimmerle believed this is possible using the classification of Schur multipliers.
REFERENCES

4. E.C. Dade. Locally trivial outer automorphisms of finite groups.