Direct design of laser-beam shapers, zoom-beam expanders, and combinations thereof

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ABSTRACT
Laser sources have become indispensable for industrial materials processing applications like surface treatment, cutting or welding to name a few examples. Many of these applications pose different requirements on the delivered laser irradiance distribution. Some applications might not only favor a specific irradiance distribution (e.g. a flat-top) but can additionally benefit from time-varying distributions.

We present an overview of a recently developed design approach that allows direct calculation of virtually any refractive or reflective laser beam shaping system. The derived analytic solution is fully described by few initial parameters and does allow an increasingly accurate calculation of all optical surfaces. Unlike other existing direct design methods for laser beam shaping, there is almost no limitation in the number of surfaces that can be calculated with this approach. This is of particular importance for optical designs of dynamic systems such as variable optical beam expanders that require four (or more) optical surfaces. Besides conventional static beam shapers, we present direct designs of zoom beam expanders, and as a novelty, a class of dynamic systems that shape and expand the input beam simultaneously. Such dynamic zoom beam shapers consist of a minimal number of optical elements and provide a much more compact solution, yet achieving excellent overall optical performance throughout the full range of zoom positions.

Keywords: Refractive laser beam shaping, Geometric optical design, Optical zoom systems, Nonimaging optics, Mathematical methods, Freeform optics

1. INTRODUCTION
Laser sources have become indispensable for industrial materials processing applications. These applications are accompanied with a variety of different demands and requirements on the delivered laser irradiance distribution. With a high spatial uniformity, flat-top beams provide benefits for applications like surface heat treatment or welding, in which it is desirable to uniformly illuminate a target surface. Other applications, like drilling and photo-polymerization can benefit from different beam profiles such as Bessel or annular beams.¹ The complexity of how to reach these imposed requirements is further increased due to number of different commercially available laser sources, ranging from 10's of watts to 10's of kilowatts.² Some applications might not only favor a specific beam irradiance distribution but can benefit additionally from time-varying distributions. The Fraunhofer institute IWS has developed a dynamic beam shaping system based on scanning mirror optics for laser beam hardening, proving a variable scan width and where the energy spread can be adapted to different local heat flow conditions.³ The use of deformable mirrors⁴,⁵ and spatial light modulators⁶ have been also proposed to achieve adaptive laser beam irradiance control. Even though adaptive concepts are very versatile, its optical efficiency might be significantly reduced and the used adaptive optics might tolerate only limited energy densities. Refractive zoom beam expanders are a well-known example and widely commercially available, e.g. for applications where the exactly required expansion ratio is not known. The basic working principle of such an optical system is illustrated in Fig. 1, consisting of three lenses (and four optical surfaces). The final beam shaping optics can be either constructed as a rotationally symmetric or a cylindrical zoom lens, depending on whether a radial magnification or magnification along one axis is needed. Besides such common zoom systems based on axial lens movements, it has been recently shown how laterally translated freeform optics can be used to design a variable-diameter flat-top beam shaping element.⁷ To our knowledge, it is the only existing example of a dynamic beam shaping system that uses movable freeform lenses.

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Figure 1. The shown movements result in a variable magnification of a collimated laser beam. Such lens profiles can be used to build a rotationally or cylindrically symmetric system.

In this work, we present an overview of a recently developed design approach that allows direct calculation of virtually any refractive or reflective laser beam shaping system. The derived analytic solution is fully described by few initial parameters and does allow an increasingly accurate calculation of all optical surfaces. Unlike other existing direct design methods for laser beam shaping, there is almost no limitation in the number of surfaces that can be calculated with this approach. This is of particular importance for optical designs of dynamic systems such as variable optical beam expanders that require four (or more) optical surfaces. Besides conventional static beam shapers, we present direct designs of zoom beam expanders, and as a novelty, a class of dynamic systems that shape and expand the input beam simultaneously. Such dynamic zoom beam shapers consist of a minimal number of optical elements and provide a much more compact solution, yet achieving excellent overall optical performance throughout the full range of zoom positions.

2. ANALYTIC SOLUTION FOR REFRACTIVE LASER BEAM EXPANDERS

Beam expanders are commonly used optical systems if the laser beam diameter has to be increased. A main function is in decreasing the divergence of the laser beam to be able to achieve controlled propagation over long distances. So called telescopic beam expanders include refractive and reflective telescopes. A refractive Galilean telescope (with a negative and a positive sub-system) which functions as a simple 2X beam expander for collimated light is shown in Fig. 2.

Figure 2. Illustration of a simple 2X laser beam expander based on two plano-aspheric lenses that doubles the size of a collimated input beam of light. In case of ideal collimation, the intensity distribution (here a flat-top) remains unchanged.
The 1st surface redistributes the rays while the 2nd surface recollimates the rays again. If ideal collimation of the input beam is assumed, the expanded intensity distribution (here a flat-top) remains unchanged. The redistribution of the rays can be described by a mapping function \( R(r) \) that describes where a ray at radial position \( r \) on the 1st surface intersects the 2nd surface at radial position \( R \).

The lens design shown in Fig. 3(b) is thus fully described by two equations for two unknown functions \( f(r) \) and \( g(R(r)) \).

![Figure 3](http://proceedings.spiedigitallibrary.org/)

(a) The mapping function \( R(r) \) of a beam expander is given by a linear function where the slope value defines the expansion factor, here 2X. (b) Introduction of all necessary initial values and functions to derive the conditional equations from Fermat’s principle.

The explicit mapping function \( R(r) \) of a 2X beam expander system is plotted in Fig. 3(a). For this simple case, the mapping is given by a linear function \( R(r) \) where the slope value \( m \) defines the magnification factor \( mX \) of the beam expander. Now that the output coordinate value \( R \) can be easily calculated for every input coordinate \( r \), it is possible to derive an integral representation for the sag of each aspherical lens that can be solved numerically following for example the work of Kreuzer, Shealy and Hoffnagle.$^{10}$

Instead of following this well-known design approach, an alternative solution strategy will be derived hereafter. As both incoming and outgoing geometric wave-fronts are plane (i.e. all rays are parallel), the two plane surfaces in Fig. 2 have no optical power and can be neglected by assuming that the in- and output rays are immersed in a medium with refractive index \( n_2 \). This is shown in Fig. 3(b). Consider an arbitrary light ray path. The ray is emitted from a point \( \vec{v}_0 \) on the plane incident wave-front and intersects the first surface at a point \( \vec{p} = (r, f(r)) \), where it is refracted towards \((R(r), g(R(r)))\) on the second surface and refracted again towards the plane outgoing wave-front (parallel to the z-axis). The path of the light rays are governed by Fermat’s principle which states that the optical path length (OPL) between two points is an extremum along the light ray containing these points. The OPL between points \( \vec{v}_0, \vec{p} \) and \((R(r), g(R(r)))\) is given by

\[
V_1 = n_2 \vec{n}_0 \cdot (\vec{p} - \vec{v}_0) + \sqrt{(r - R(r))^2 + (f(r) - g(R(r)))^2}
\]  
(1)

where \( \vec{n}_0 \) denotes the directional vector of the wave-front in z-direction. If the points \( \vec{v}_0 \) and \((R(r), g(R(r)))\) are fixed, the only free parameter that can vary to achieve an extremum for \( V_1 \) is the point \( \vec{p} \) and hence \( r \) on the first lens surface. Fermat’s principle thus implies that

\[
D_1 = \frac{\partial}{\partial r} V_1 = 0
\]  
(2)

where the partial derivative indicates that \( R \) is held fixed. Furthermore, the slope value on the first aspheric surface at point \((r, f(r))\) must be equal to the slope value on the second aspheric surface at point \((R, g(R))\) as a result of symmetry of the input and output wave-fronts. This can be expressed in a second condition

\[
D_2 = f'(r) - \partial_r g(r) \big|_{r=R(r)} = 0
\]  
(3)

where the partial derivative \( \partial_r g(r) \) is evaluated at position \( R(r) \). This specific formulation highlights the functional character of the optical design problem. As the mapping function \( R \) is a function of \( r \) and appears as an argument in the unknown function \( g \), Eqs. (2) and (3) form a pair of functional differential equations. The lens design shown in Fig. 3(b) is thus fully described by two equations for two unknown functions \( f(r) \) and \( g(R(r)) \).
Suppose that \((f, g)\) is an analytic and smooth solution to the functional differential equations (2) and (3), Taylor’s theorem implies that the functions must be infinitely differentiable and have a power-series representation

\[
f(r) = \sum_{i=0}^{\infty} f_i (r - r_0)^i \quad g(r) = \sum_{i=0}^{\infty} g_i (r - R(r_0))^i
\]

centered at \((r_0, z_0)\) and \((R(r_0), z_1)\), respectively. Due to symmetry, the initial conditions for an on-axis ray are given by the following function values and first derivatives at \(r = 0\)

\[
f(0) = 0 \quad f'(0) = 0 \quad g(0) = d \quad g'(0) = 0
\]

also indicated by "ray 1" in Fig. 3(b). The initial values satisfy the conditional equations \(D_i = 0\) for \(i = 1, 2\) and provide general solutions for the initial Taylor series coefficients \(f_0, f_1, g_0\) and \(g_1\), similar to our previous work. In ascending order, it is now possible to calculate \((n+1)^{th}\) order Taylor polynomials for \(f(r)\) and \(g(r)\) by evaluating equations

\[
\lim_{r \to r_0} \frac{\partial^n}{\partial r^n} D_i = 0 \quad (i = 1, 2), \quad \{n \in \mathbb{N}_0\}.
\]

at position \(r_0\). The case \(n = 0\) corresponds to the just solved equations for the initial Taylor series coefficients. For \(n \geq 1\), the set of equations (6) results in two linear algebraic equations for particular Taylor series coefficients \(f_{n+1}\) and \(g_{n+1}\). The elements of the linear algebraic equations consist of mathematical expressions only dependent on previously calculated Taylor series coefficients. Therefore, all Taylor series coefficients can be calculated as exact symbolic expressions in ascending order - step by step. So far, no approximations have been made. The general solution scheme for \(n \geq 1\) allows to calculate successive Taylor series coefficients of \(f(r)\) and \(g(r)\) as symbolic solutions up to a very high order (\(> 20^{th}\)). The inevitable truncation of the infinite sum of terms will be the only approximation made.

### 2.1 Evaluation of the analytic solution

The symbolic calculations of the Taylor series coefficients of the analytic solution are implemented as a function in Wolfram Mathematica. To evaluate the performance of different solutions of a certain order \(n\), 100 equidistant rays ranging from 0 mm to 2 mm (first lens aperture) along the \(r\)-axis are traced and evaluated in Mathematica. The spatial deviation of each ray at \(r_i\) is then defined by the mapping error \(R_i' - R(r_i)\). Where \(R_i'\) is the calculated intersection of the \(i^{th}\) ray with the surface profile \(g\) and \(R(r_i)\) is the ideal position defined by the mapping function. The angular deviation of each ray is given by the directional cosine \(\cos \theta_r\) in \(r\)-direction after the refraction at \(g(R_i')\), which should be zero in the ideal case. The evaluation is performed for \(4^{th}\) \((n=4)\), \(6^{th}\) \((n=6)\) and \(8^{th}\) \((n=8)\) order Taylor polynomials for \(f\) and \(g\), respectively. Figure 4(a) shows the collimation error and Fig. 4(b) shows the spatial mapping error for the 3 calculated solutions of different order.

![Figure 4](http://proceedings.spiedigitallibrary.org/)

\[\text{Figure 4. Demonstration of the convergence of the analytic solution: (a) the collimation error and (b) the spatial mapping error clearly reduce with an increasing approximation order.}\]

For both graphs, the collimation and mapping error reduce very well with an increasing order underlining the convergence and achievable high accuracy of the derived solution. The overall root mean square values of the
collimation error are $0.729 \times 10^{-3}$ (n=4), $0.206 \times 10^{-3}$ (n=6) and $0.047 \times 10^{-3}$ (n=8). The root mean square values of the mapping error are $1.398 \times 10^{-2}$ (n=4), $0.324 \times 10^{-2}$ (n=6) and $0.080 \times 10^{-2}$ (n=8), respectively. Further accuracy improvements, if required, can be achieved by just increasing the order of the Taylor polynomials.

3. ANALYTIC SOLUTION FOR GAUSSIAN TO FLAT-TOP BEAM SHAPERS

A very common design task is to take the Gaussian irradiance distribution produced by a laser and transform it into a flat-top output. The input distribution is given by the Gaussian function

$$I_{in}(r) = \frac{2}{\pi w_0^2} \cdot \exp \left[ -2 \left( \frac{r}{w_0} \right)^2 \right]$$  \hspace{1cm} (7)

with $1/e^2$ width $w_0$. To avoid diffraction effects in regions of abrupt irradiance change and to increase the distance over which the uniform irradiance distribution can be used, \(^1\) the output intensity is chosen as a flattened Lorentzian (FL) distribution, which was introduced by Brenner in 2003, \(^1\)

$$I_{FL}(R) = \frac{1}{\pi R_{FL}^2 (1 + (R/R_{FL})^q)^{1+\frac{2}{q}}}$$  \hspace{1cm} (8)

where the profile has a flat central region and a gradual roll-off to the null region. The value $q$ denotes the beam shape parameter and $R_{FL}$ is the beam width parameter of the flattened Lorentzian. The input encircled energy is calculated as a function of $r$ as

$$A(r) = \int_0^r I_{in}(r') 2\pi r' \, dr' = 1 - \exp \left[ -2 \left( \frac{r}{w_0} \right)^2 \right]$$  \hspace{1cm} (9)

and the output encircled energy as a function of $R$ as

$$B(R) = \int_0^R I_{FL}(R') 2\pi R' \, dR' = \left[ 1 + (R/R_{FL})^{-q} \right]^{-(2/q)}$$  \hspace{1cm} (10)

From conservation of energy between the input and the output plane, it is possible to determine the output radial distance $R$ as a function of the given input coordinate $r$ such that the encircled energies $A$ and $B$ are equal, that is

$$R(r) = \frac{\epsilon R_{FL} \sqrt{1 - \exp \left[ -2 \left( \frac{r}{w_0} \right)^2 \right]}}{\sqrt[4]{1 - \left[ 1 - \exp \left[ -2 \left( \frac{r}{w_0} \right)^2 \right] \right]}}^{q/2}$$  \hspace{1cm} (11)

where $\epsilon$ is equal to $+1$ for a Galilean and $-1$ for a Keplerian configuration. \(^1\) An explicit mapping function $R(r)$ that transforms a Gaussian into a flattened Lorentzian irradiance distribution is shown as dotted line in Fig. 5(a) for $\epsilon = 1$, $w_0 = 2.366$ mm, $R_{FL} = 3.25$ mm and $q = 16$.

The analytic solution to convert a Gaussian to a flattened Lorentzian intensity distribution works very similar to the just solved beam expander case. However, few fundamental differences do exist. The full description of the necessary solution steps are given in our previous work, \(^8\) and will only be summarized here:

As it turned out, a one-point Taylor series expansion does not converge for the mapping function \((11)\), which is shown in Fig. 5(a). Therefore, a Taylor series expansion about two points $r_1$ and $r_2$ can be defined as

$$T(r) = \sum_{i=0}^{k} \left[ a_i (r - r_1) + b_i (r - r_2) \right] [(r - r_1)(r - r_2)]^i$$  \hspace{1cm} (12)
which is a \((2k + 1)\)th degree polynomial approximation of the mapping function \(R(r)\), if the coefficients \(a_i\) and \(b_i\) do fulfill that \(\partial^i T/\partial r^i\) is equal to \(\partial^i R/\partial r^i\) at \(r_1\) and \(r_2\) for \(i = 0, 1, \ldots, k\). This two-point Taylor series approximation of the mapping function converges well for all values \(r\) and is added to the graph in Fig. 5(a).

The selection of points \(r_1\) and \(r_2\) depends strongly on the shape of the considered mapping function. To find a suitable pair of points, it has been found sufficient to verify that the two-point Taylor series approximation of the mapping function converges well for all values \(r\). This was always the case even for manually selected \(r_1\) values close to the optical axis and \(r_2\) values close to the maximum radius. There is no strict mathematical proof that the same convergence also applies for the lens profiles. However, all evaluated designs demonstrated this behavior so far. The results presented in this work will provide strong evidence that the solutions for the lens profiles are analytic and smooth as well. Therefore, the surface profile functions can be expressed by two-point Taylor series expansions

\[
f(r) = \sum_{i=0}^{k} \left[ p_i (r - r_1) + q_i (r - r_2) \right] \left[ (r - r_1)(r - r_2) \right]^i
\]

\[
g(r) = \sum_{i=0}^{k} \left[ u_i (r - r_3) + v_i (r - r_4) \right] \left[ (r - r_3)(r - r_4) \right]^i
\]

about points \((r_1, z_1)\) and \((r_2, z_2)\), and \((r_3, z_3)\) and \((r_4, z_4)\) respectively. The paths of both corresponding rays are shown in Fig. 5(b). For the first ray, \(z_1\) can be freely chosen without loss of generality, whereas the value \(z_3\) fixes the distance between the two lens surfaces. For the second ray, the position \(z_2\) can not be determined analytically and represents a single parameter that links the two local solutions at \(r_1\) and \(r_2\). The position \(z_4\) then follows from a constant optical path length condition. All four slope values \(m_1, m_2, m_3\) and \(m_4\) at points \(r_i\) \((i = 1..4)\) follow from the ray mapping relationship and Snell’s law. In ascending order, it is now possible to calculate \((2k+1)\)th order Taylor coefficients of \(f(r)\) and \(g(r)\) by calculating the derivatives and evaluating

\[
\lim_{r \to r_1} \frac{\partial^n}{\partial r^n} D_i = 0, \quad \lim_{r \to r_2} \frac{\partial^n}{\partial r^n} D_i = 0 \quad \text{for } (i = 1, 2), \{n \in \mathbb{N}_1\}
\]

symbolically at points \(r_1\) and \(r_2\). The case \(n = 0\) then solves the equations for the initial Taylor series coefficients. For \(n \geq 1\), the set of equations (15) results in four linear algebraic equations for particular two-point Taylor series coefficients \(p_i, q_i, u_i, v_i\) that can be solved analytically. The general solution scheme for \(n \geq 1\) allows to calculate successive Taylor series coefficients of \(f(r)\) and \(g(r)\) up to a very high order. As before, the truncation of the infinite sum of terms is the only approximation made.

### 3.1 Evaluation of the analytic solution

The performance evaluation of the analytic solution for Gaussian to flattened Lorentzian (FL) beam shaping is carried out analogously to Sec. 2.1. The parameters of the input and output intensity distributions are
w_0 = 2.366 mm, R_{FL} = 3.25 mm and q = 16. The two manually selected points of the Taylor series are r_1 = 0.2 mm and r_2 = 3.6 mm. By fixing the distance between z_1 = 0 mm and z_3 = 16 mm and the refractive index n_2 = 1.5, the remaining parameter z_2 adjusts the relative positioning in z-direction and is calculated using a local optimization algorithm in Mathematica. To quantitatively evaluate the accuracy of the derived solution, the spatial and angular deviation is calculated for 100 rays between 0 and 4 mm along r-direction. This evaluation is performed for 7th (k=3), 11th (k=5) and 15th (k=7) order Taylor polynomials for f and g, respectively. Figure 6(a) shows the collimation error and Fig. 6(b) shows the spatial mapping error for the 3 calculated solutions of different order.

![Figure 6](http://example.com/fig6.png)

Figure 6. Demonstration of the convergence of the analytic solution: (a) the collimation error and (b) the spatial mapping error clearly reduce with an increasing approximation order.

The root mean square values of the collimation error are 1.586 x 10^{-3} (k=3), 0.563 x 10^{-3} (k=5) and 0.372 x 10^{-3} (k=7). The root mean square values of the mapping error are 0.585 x 10^{-2} (k=3), 0.102 x 10^{-2} (k=5) and 0.020 x 10^{-2} (k=7), respectively. For both graphs, the collimation and mapping error reduce very well with an increasing order underlining the convergence and achievable high accuracy of the derived solution. Further improvements, if necessary, can be achieved by further increasing the order of the Taylor polynomials and/or by fine-tuning the positions of points r_1 and r_2, respectively.

4. FREEFORM OPTICAL DESIGN OF AN XY-ZOOM BEAM EXPANDER

In this Section, we present a novel zoom XY-beam expander based on freeform optics and axial movements. Such an optical system allows to simultaneously vary the magnification in x- and y-direction, respectively. This conceptual idea itself is not new and has been for example proposed by Melles Griot researchers as an XY-beam expander for high-power UV laser beam shaping. Their system consists of five elements, two moving x-oriented cylindrical, one fixed spherical, and two moving y-oriented cylindrical lenses. The new in this work is that axially moving freeform lenses are used to achieve such an optical functionality: using less optical elements, building more compact optical systems, while achieving excellent overall optical performance.

The optical design approach that is presented in this section is loosely based on the original optical design method described in. The original design method has been already successfully extended for designing static laser beam shaping systems consisting of two aspherical lenses by incorporating the ray mapping relationship. The main argument for developing this novel design approach for laser beam shaping has been the fact that we have already demonstrated the feasibility to incorporate relative lens movements in the mathematical description to design a dynamic solar concentrator system in the past.

The core idea and basic concept of such a variable beam expander in x- and y-direction based on freeform lenses is illustrated in Fig. 7. The optical system consists of three elements, whereas the first and last lens have each one plane surface. In the most general case, all three lenses can be allowed to move relatively to each other. Furthermore, for an afocal system, both incoming and outgoing geometric wave-fronts should be plane (i.e. all rays are parallel). This means that the plane entry and exit surfaces in Fig. 7 have no optical power and can be neglected, assuming that the in- and output rays are immersed in the lens material with refractive index n_2.

It is well-known from literature, that two optical surfaces are sufficient to design rotationally symmetric or freeform laser beam shapers. The common design approach is based on the geometric ray mapping relationship which describes where an initial ray that is refracted at the first surface will be redirected to and re-collimated at
Figure 7. Introduction of all required initial values and functions to derive the conditional equations from Fermat’s principle. The XY-zoom beam expander design is described by two different lens configurations (a), (b) and (c), (d) along the x- and y-axis, respectively.

The second surface. This design method is limited to two optical surfaces and does not allow to design dynamic beam shaping systems. In case of a basic beam expander, the ray mapping relationship is characterized by a linear function, where the slope value defines the magnification. By using four shaped optical surfaces, e.g. in Figs. 7(a) and 7(c), it is reasonable to assume that it should be possible to design the lens profiles to fulfill different magnification values if the lenses are shifted (in a specific way). Such a design, if the lens profiles are rotated about the common optical axis, would then perform as a conventional zoom beam expander.

To our knowledge, none of the existing direct design methods are currently capable of designing such rotationally symmetric systems that consist of more than 2 surfaces and where the lenses are movable at the same time. An additional novelty of the shown optical design is the fact that besides the zoom magnifications along the x-axis, the system is supposed to provide different magnifications along the y-axis, while the lenses are moved in the same way, shown in Figs. 7(b) and 7(d). The freeform lens surfaces can be described by four functions $e(x, y), f(x, y), g(x, y)$ and $h(x, y)$. The different magnifications along two orthogonal axes mean that the problem is separable and solutions along the x- and y-axis can be independently obtained. However, the fact that all lens profiles along the x- and y-axis should obviously share common lens vertices introduces a relationship between the otherwise independent sub-solutions. The relative displacements of the lenses in configuration 2 with respect to configuration 1 are introduced through two off-set values $z_1$ and $z_2$. From now on, we will use a simplified notation for the independent lens profiles $e(x) = e(x, 0)$ and $e(y) = e(0, y)$, and accordingly for $f, g$ and $h$.

Let’s consider the design problem along the x-axis in Figs. 7(a) and 7(c) (the y-axis solution then works exactly the same). An arbitrary ray path through the optical system for the 1st configuration in (a) can be described by a sequence of points $(x, W_0), (x, e(x)), (s(x), f(s(x)), (t(x), g(t(x)), (m_1(x), h(m_1(x))) and (m_1(x), W_1))$: where the auxiliary functions $s(x)$ and $t(x)$ describe where the ray intersects with the lens surfaces $f$ and $g$ as a function of $x$. Finally, as intended by this optical design, the ray intersects the last surface $h$ at an x-coordinate value which is given by the (linear) ray mapping relationship $m_1(x)$. The values $W_0$ and $W_1$ denote arbitrary but fixed
positions in z-direction of reference points on both incoming and outgoing plane wave-fronts. The overall optical path length of a ray can then be expressed in sections \( d_1, \ldots, d_5 \) using vector geometry.

\[
\begin{align*}
    d_1 & = -n_2(W_0 - e(x)) \\
    d_2 & = \sqrt{(e(x) - f(s))^2 + (x-s)^2} \\
    d_3 & = n_2(\sqrt{(f(s) - g(t))^2 + (s-t)^2} \\
    d_4 & = \sqrt{g(t) - h(m_1)^2 + (t-m_1)^2} \\
    d_5 & = n_2(h(m_1) - W_1) \\
    d_6 & = -n_2(W_0 - e(x)) \quad (16) \\
    d_7 & = \sqrt{(e(x) - f(u) - z_1)^2 + (x-u)^2} \quad (17) \\
    d_8 & = n_2(\sqrt{(f(u) - g(v))^2 + (u-v)^2} \quad (18) \\
    d_9 & = \sqrt{(g(v) + z_1 - h(m_2) - z_2)^2 + (v-m_2)^2} \quad (19) \\
    d_{10} & = n_2(h(m_2) + z_2 - W_1) \quad (20)
\end{align*}
\]

Similarly, the optical path of an arbitrary ray through the optical system for the 2nd configuration in (c) can be expressed in sections \( d_6, \ldots, d_{10} \). As before, the auxiliary functions \( u(x) \) and \( v(x) \) describe intermediate ray positions on surfaces \( f \) and \( g \), the values \( z_1 \) and \( z_2 \) define shifts of the lenses, and the linear ray mapping relationship \( m_2(x) \) defines where the ray intersects the last surface \( h \). All four lenses consist of the same material with refractive index \( n_2 \) at a certain wavelength. Fermat’s principle states that a light ray between two fixed points will follow the path for which the optical path length is an extremum. Consider two arbitrary but fixed points on the wave-front defined by \((x, W_0)\), and \((s(x), f(s(x)))\) on the second lens profile: a ray coming from the wave-front and passing through \((s(x), f(s(x)))\) must be such that the combined optical path length \( d_1 + d_2 \) is an extremum. With both end points kept fixed, the only remaining variable to achieve an extremum for \( d_1 + d_2 \) is the point \((x,e(x))\) on the first lens profile. Fermat’s principle implies that \( D_1 = \frac{\partial}{\partial x_i}(d_1 + d_2) = 0 \), where the partial derivative indicates that the point \((s, g(s))\) is kept fixed. By applying Fermat’s principle to all neighboring optical path length sections \( d_1, \ldots, d_5 \) by pairs, it is possible to derive four functional differential equations

\[
D_i = \frac{\partial}{\partial x_i}(d_i + d_{i+1}) = 0 \quad (i = 1.4), \quad \chi = \{x, s, t, m_1\}.
\]

for the 1st configuration in Fig. 7(a). In a similar manner, it is possible to derive four additional functional differential equations

\[
D_{i-1} = \frac{\partial}{\partial x_i}(d_i + d_{i+1}) = 0 \quad (i = 6..9), \quad \chi = \{x, u, v, m_2\}.
\]

for the 2nd configuration in Fig. 7(c). The lens design along the x-axis is thus fully described by eight functional differential equations \( D_1, \ldots, D_8 \) for four unknown surface profiles \( e, f, g \) and \( h \) and four unknown auxiliary functions \( s, t, u \) and \( v \). These differential equations cannot be solved explicitly, thus a Taylor series method is used to solve them. Suppose that the unknown functions are analytic and smooth and a solution to the functional differential equations, Taylor’s theorem thus implies that the functions must be infinitely differentiable and have a power-series representation. Thus the eight functions can be expressed by power series

\[
\begin{align*}
    e(x) & = \sum_{i=0}^{\infty} e_i x^i \\
    f(x) & = \sum_{i=0}^{\infty} f_i x^i \\
    g(x) & = \sum_{i=0}^{\infty} g_i x^i \\
    h(x) & = \sum_{i=0}^{\infty} h_i x^i \\
    s(x) & = \sum_{i=0}^{\infty} s_i x^i \\
    t(x) & = \sum_{i=0}^{\infty} t_i x^i \\
    u(x) & = \sum_{i=0}^{\infty} u_i x^i \\
    v(x) & = \sum_{i=0}^{\infty} v_i x^i \\
\end{align*}
\]

centered at the optical axis. The coefficients can be calculated by substituting the power series functions into the differential equations, assuming that all higher order derivatives do exist at \( x = 0 \) where the substituted differential equations are evaluated. Now it is possible to calculate \((n+1)\)th order Taylor series coefficients for the surfaces and \( n \)th order for the auxiliary functions by evaluating and solving equations

\[
\lim_{x\to 0} \frac{\partial^n}{\partial x^n} D_i = 0 \quad (i = 1.8), \quad \{n \in \mathbb{N}_1\}.
\]

in ascending order. The composite functions \((f(s(x)), g(t(x)), \ldots)\) and their derivatives are first evaluated for the auxiliary functions \((s(x), t(x), \ldots)\) for \( x \to 0 \) and then inserted into the respective function \((f(x), g(x), \ldots)\) for \( x \to 0 \). For \( n = 0 \), Eqs. (25) are satisfied if all first derivatives are equal to zero, i.e. \( e_1, f_1, g_1 \) and \( h_1 \) are zero; and the auxiliary functions coefficients \( s_0, t_0, u_0 \) and \( v_0 \) are zero. This is the case where the on-axis ray coincides with the optical axis.
1. For \( n = 1 \), Eqs. (25) result in nonlinear algebraic equations for Taylor series coefficients \( e_2, f_2, g_2, h_2, s_1, t_1, u_1 \) and \( v_1 \). These equations have been solved using MATLAB’s fsolve function (any other solver for systems of nonlinear equations could be used).

2. For \( n > 1 \), Eqs. (25) result in a system of linear equations for Taylor series coefficients \( e_{n+1}, f_{n+1}, g_{n+1}, h_{n+1}, s_n, t_n, u_n \) and \( v_n \); where the coefficients of the system and the constant terms are mathematical expressions dependent on previously calculated Taylor series coefficients of lower order. Thus, it is possible to calculate the Taylor series coefficients in ascending order for \( n = 2, 3, 4, \ldots \) by solving each time the linear system of equations in Eqs. (25) using a Gaussian elimination algorithm.

So far, no approximations have been made. The solution scheme allows to calculate the Taylor series coefficients of \( e(x) f(x), g(x), h(x) \) and the auxiliary functions up to very high orders. The calculated finite Taylor polynomials are truncated sums of the exact (infinite) series, and will be the only approximation made. The radii of convergence for functions \( e(x), f(x), g(x) \) and \( h(x) \) are important, as they indicate the maximum lens apertures that can be achieved. Taylor’s remainder theorem provides quantitative estimates for the approximation error but it can be also verified directly via ray tracing simulations. The symbolic solutions for the (non)-linear equations for \( n \geq 1 \) have been calculated in Mathematica, then automatically translated to C++ code and finally embedded in a MATLAB-compatible .mex file library. Once compiled, this library returns the Taylor polynomial coefficients for all lens profiles up to 16th order, and the auxiliary functions up to 15th order, along either the x- or y-axis, for given initial values. It is important to point out that the solution scheme does not allow to determine a priori for which initial values simultaneous solutions in x and y do exist and converge well. The ray tracing results presented in the following section will provide evidence that such solutions do exist.

### 4.1. Evaluation of the continuously operating zoom system

The optical simulations in this section have been done using a self-made analytic ray tracer in MATLAB that uses the fzero function to calculate ray surface intersections. The five remaining initial values of the lens vertices (lens thicknesses) and lens offset values (relative movement) have been automatically adjusted using MATLAB’s fminsearch optimization in order to minimize the re-collimation error of the traced rays for both configurations at the exit plane in Fig. 7. The evaluation of one quadrant is sufficient due to the overall symmetry of the optical design. The used merit function is given by the sum of the absolute values of the direction cosine with respect to the optical axis for 25 (5x5) traced rays on an equidistant square grid ranging from 0 to 3 mm along x- and y-direction, respectively. As boundary conditions, the length of the optical system (distance between \( e_0 \) and \( h_0 \)) has been limited to 100 mm, and the thickness of the central lens (distance between \( f_0 \) and \( g_0 \)) to 6 to 8 mm. The optimization of the initial parameters converges quickly and a solution was obtained for values \((f_0, g_0, h_0, z_1, z_2) = (-85.71, -92.70, -97.66, 79.56, -5.96) \) mm and the detector placed at \( z = 120 \) mm. Figure 8 shows the corresponding optical path difference (OPD) or geometrical wave-front error evaluated for one quadrant of the detector plane, for both design configurations and an equidistant grid of 400 traced rays.

![Optical path difference / \( \lambda \)](image)

---

Figure 8. The shown low wave-front errors confirm the high accuracy of the simultaneously calculated solutions for configuration 1 (left) and 2 (right).
As an alternative, the same results can be obtained when the variation in optical path length is minimized for all rays. As it can be already expected from the cross sections shown in Fig. 7, all four freeform surfaces share saddle points at the optical axis, which is visualized in the contour plots in Fig. 9.

Figure 9. The surface contour plots of the calculated freeform surfaces show and confirm that all four surfaces share saddle points at the optical axis.

These low errors clearly confirm that freeform optical solutions for simultaneous XY-beam expansions in two configurations do exist and that the analytic solution scheme that has been proposed in Sec. 4 can provide accurate solutions to the stated optical design problem. If desired, an even higher accuracy can be achieved by either increasing the order of the calculated polynomials and/or increasing the overall length of the optical system. It is also important to emphasize that either of the analytic sub-solutions along the x- or y-axis could be directly used to design conventional rotationally symmetric zoom beam expander systems.

After the validity of the derived analytic solution for the two design configurations has been confirmed, the next step is the evaluation of the in-between performance when the lenses are allowed to move continuously. First, it is necessary to determine the trajectories of the lens’ positions that will provide a continuously operating zoom system. One possible way could be to calculate the mutual axial distances between individual lens elements based on a paraxial analysis, as proposed in.\textsuperscript{19} In contrast to these direct calculations, the positions of the lenses have been determined using a basic optimization approach: the magnification along the x-axis has been scanned through from values \(0.5 \times\) (configuration 2) to \(2 \times\) (configuration 1) in 25 steps. In addition, the desired magnification values along the y-axis have been set to be the inverse value of the magnification in x. For each step, the lens offset values \(z_1\) and \(z_2\) have been optimized for a merit function as defined in the previous Section, plus a second figure of merit, given by the sum of the root mean square values of the ray mapping error of the rays. This spatial deviation is defined as the distance between the real ray position at the detector and the ideal ray position according to the targeted magnifications in x and y, with a constant detector position at \(z = 120\) mm.
With the known approximated lens trajectories, it is now possible to continuously scan through the full magnification range. This has been done for the magnification along the x-axis ranging from 0.5× to 2× in 150 steps. The ray tracing has been done using the software ASAP and tracing 2,000,000 rays for each step. The lenses have been implemented as analytic Taylor polynomials and shifted according to the approximated lens trajectories. As an input beam, an ideally collimated rectangular grid source has been chosen that fills the complete clear aperture of the first lens (6×6 mm²). A specific intensity distribution can be defined using an apodization function in ASAP. The used intensity distribution for this example here is given by the product of two flattened Lorentzian\(^{12}\) (FL) distributions in x and y, where \(R_{FL}=2\) is the FL width parameter, \(q=16\) is the FL shape parameter.\(^{10}\) The square-shaped distribution has a flat central region and a gradual roll-off to the null region. The constant \(I_0\) is chosen in such a way so that the flat central intensity at the detector fluctuates around 0.9 W/mm\(^2\) and no saturation effects appear when an identical color bar ranging from 0 to 1 W/mm\(^2\) is used for all simulation steps. It should be noted that any other input distribution (e.g., a Gaussian or circular flat-top distribution) could be used to demonstrate the functionality of the designed optical system. The main argument for a square-shaped flat-top input distribution is the fact that it is much easier to visually validate the optical performance than it would be the case, for example, for any circular input intensity distribution. Figure 10 shows three different zoom stages as well as the obtained intensity distribution at the detector.

<table>
<thead>
<tr>
<th>x-z plane cross section</th>
<th>y-z plane cross section</th>
<th>Intensity at detector</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st zoom stage</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2nd zoom stage</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3rd zoom stage</td>
<td></td>
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</tbody>
</table>

Figure 10. Different zoom stages of an XY beam expander system with variable changes of the aspect ratio of the intensity distribution at the detector.

The intermediate zoom stage confirms a very good optical performance of the system for all in-between zoom positions, even though the optical design has been only done for two discrete states, namely configuration 1 and 2. This can be observed very clearly as the intensity remains flat-top also for the second zoom stage. It should be emphasized that the achieved intensity distributions do not remain unchanged for the entire zoom range in case the detector position is altered. This is especially the case for zoom positions far off the two design configurations. Improving the overall performance should be possible if more than four lens surfaces will be used; similar to most commercial (circular symmetric) zoom beam expanders that consist of 4 or more lenses (with even more optical surfaces). From an optical design method point of view, this could be realized by adding one (or more) additional configuration(s), the mathematical approach would remain the same. In case of six (eight) optical surfaces, a third (fourth) in-between configuration(s) could be added which would not only guarantee near-ideal performance towards the two ends of the full zoom range but also at one (two) zoom position(s) better covering the central zoom region. A final optimization can be used to further balance the optical performance throughout the full zoom range; but also to take a realistic (non-zero) beam divergence of the beam into account.
5. FREEFORM OPTICAL DESIGN OF COMBINED ZOOM BEAM SHAPERS

In this Section, we present an optical zoom system that consists of three freeform lenses (two with one plane and one freeform surface, one with two freeform surfaces) that are movable along one common optical axis. The surface shapes are calculated to generate an afocal zoom system that transforms a given input Gaussian beam to squared flat-top output energy distributions of variable size. Such an optical performance can be already achieved using known system solutions. First, a collimated beam shaping system (two groups of lenses) is used to shape a Gaussian beam to a rectangular square. As second sub-system, a common afocal zoom beam expander (three groups of lenses) is used now to scale the size of the generated rectangular square throughout the zoom range. In contrast, the design example presented here combines both functionalities in a single afocal zoom system, using less optical elements and, hence, providing a more compact optical system.

As the solution in Sec. 3 does not converge over the full aperture of the lenses, it is not directly possible to solve the combined zoom beam shaper exactly as shown in Sec. 4. However, it is possible to use the XY-zoom beam expander design concept and solve it repeatably for different magnifications each time and a Gaussian to flattened Lorentzian mapping function - the same in x and y (to have a squared output distribution). This step-wise approach makes use of the fact that the solution converges in a certain neighborhood of the central point of the Taylor series (compare Fig. 5(a)). The first solution is obtained in the vicinity of the optical axis. Next, x and y are increased by a step ∆d, and the calculation is repeated using the results from the previous step. The step-size does not need to be very small, it is only important to ensure that the previous step was still converging at the new step position. This step-wise design calculation is repeated until the full lens apertures are solved. The final solution is then a combination of point clouds that can be fitted with an XY polynomial functions, resulting in the zoom system shown in Fig. 11.

Figure 11. Different zoom stages of an zoom beam shaper system that transforms a Gaussian to a flat-top squared distribution of variable size at the reference plane.
The designed example is based on PMMA lenses with a refractive index of 1.48 at the wavelength 632.8 nm. The input beam is a collimated Gaussian distribution with waist diameter of 2.366 mm. The system magnifies the input beam by a factor of 1x, 2x and 3x both in x and y for the shown three configurations. A final optimization in Zemax has been performed in order to balance both the shaping and collimation quality at the output for all three zoom stages.

6. CONCLUSION

Within the scope of this work, it has been demonstrated how the optical design of refractive laser beam shaping systems based on two plano-aspheric lenses can be described by means of a new mathematical formulation using functional differential equations. The derived solution scheme has been successfully verified for three specific design tasks and allows an increasingly accurate calculation of the lens profiles described by high order Taylor polynomials. In case of the flattened Lorentzian beam shaping system, an RMS collimation error of 0.372 mrad has been achieved. If required, a higher accuracy seems feasible with this new design approach. Furthermore, it has been shown how axially moving freeform lenses can be used to achieve a simultaneously varying magnification both in x- and y-direction, respectively. In comparison with (existing) combinations of rotated cylindrically symmetric zoom beam expanders, such a freeform system consists of less optical elements and provides a much more compact solution, yet achieving excellent overall optical performance. As a further generalization of this method, the step-wise calculations illustrated in Sec. 5 show how arbitrary (non-linear) mapping functions can be used to include additional optical functionalities like simultaneous beam shaping in the movable freeform zoom lens design.

The presented analytic solution is fully described by very few initial parameters and does allow an increasingly accurate calculation of the lens profiles described by high order Taylor polynomials. The description of the lens surfaces with analytic functions allows imposing boundary conditions (e.g., on the local curvature of the surfaces) directly in the design process to meet manufacturing requirements. Moreover, this solution approach can be adapted to cope with additional optical surfaces and/or lens groups to further enhance the overall optical performance. A final optimization of the calculated lens surfaces could prove beneficial to then further balance the performance throughout the full range of continuous zoom positions. Last but not least, the presented sub-solutions along one of the coordinate axes can be directly used to design rotationally symmetric zoom laser beam expanders and/or shapers without the limitation to only two optical surfaces. This additional functionality makes the here presented direct design approach widely applicable to zoom beam expander systems in general. In cases where a re-collimation of the laser beam is not a requirement but only a fixed target area needs to be illuminated, the design approach presented in this paper could prove beneficial as well. A very well suited application example for such a case is a dynamic beam delivery system for laser beam hardening, where the here proposed freeform optical system would be capable of illuminating a target region with a variable aspect ratio. Identifying the most promising application areas and manufacturing a proof-of-concept demonstrator for experimental validation is part of our planned future work.

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