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Achievability of the Rate-Distortion Function in Binary Uniform Source Coding With Side Information

Andrei Sechelea∗‡, Adrian Munteanu∗‡, Aleksandra Pizurica†‡, and Nikos Deligiannis∗‡

∗Department of Electronics and Informatics, Vrije Universiteit Brussel, Brussels, Belgium
†Department for Telecommunications and Information Processing, Ghent University, Gent, Belgium
‡iMinds, Gaston Crommenlaan 8(b102), 9050 Gent, Belgium

E-mail: {atsechel, acmuntea, ndeligia}@etro.vub.ac.be, Aleksandra.Pizurica@telin.ugent.be

Abstract—In this work we consider asymmetric correlation channels and focus on the achievability of the rate-distortion function when the source is binary uniform. This lies in contrast with conventional symmetric correlation channel models employed in the literature. The rate-distortion function for binary source coding with correlated side information is expressed as a function of an auxiliary random variable with an alphabet size smaller or equal to three. Complementing our recent findings on the problem, we propose a numerical analysis of the differences between assuming binary and ternary auxiliary variables in the derivation of the rate distortion bound. We also show that, even if the proposed bound does not admit an analytical expression, it always admits a unique solution.

Index Terms—Source coding with side information, Wyner-Ziv coding, rate-distortion function

I. INTRODUCTION

Low-power wireless network applications, such as wireless sensor networks [1], [2], wireless capsule endoscopy [3], and visual sensor networks [4], have promoted a coding framework in which multiple correlated sources are independently compressed, transmitted and jointly decoded at a central processing unit. The fundamentals of this coding framework, which is known as distributed source coding (DSC), were set by Slepian and Wolf [5], who considered separate lossless compression and joint decoding of two correlated information sources. They have shown that a total rate equal to the joint entropy is sufficient to achieve lossless compression, when the correlation between the sources is only used at the decoder. In [6], Wyner and Ziv analyze the specific case when the decoder has access to an additional information source, correlated with the input, termed side information.

We distinguish two scenarios of source coding with side information, as illustrated in Fig. 1. The first one is the conventional predictive case, with the side information available both to the encoder and the decoder. In Fig.1, this corresponds to both switches I and II being closed.

Fig. 1. Schematic illustration of source coding with side information setups.

The second case is the Wyner-Ziv setup, when the side information is only available at the decoder. In Fig.1, this corresponds to the case when switch I is open, while switch II is closed. The large majority of the techniques in discrete source coding with side information consider a symmetric correlation between source and side information. Nevertheless, recent results in distributed video coding [3], [7]–[9] show that asymmetric correlation models provide consistent gains over symmetric models [10], [11].

Historically, the binary symmetric correlation was the first case to be studied, owing to its simplicity. The rate distortion functions were given by Berger in [12] for the predictive case, and respectively by Wyner and Ziv in [6], for the corresponding WZ case. Another binary correlation channel that has been recently considered is the Z-channel, for which Steinberg derived in [13] the rate distortion bound \( R_{Z}^{X|Y}(D) \) for the predictive case, while in [14] we derived the rate distortion bound \( R_{WZ}^{X|Y}(D) \) for the Wyner-Ziv case. In [6], it is established that, for the general case, a rate loss exists between the predictive and WZ bounds. Zamir showed in [15] that, for binary sources and the Hamming distortion metric, the rate loss is at most 0.22 bits/sample, and in [14] we show that this difference vanishes in the case of the Z-channel correlation.

However, the formulation of the binary rate distortion function for the general correlation case was not available
in the literature. As such, we have presented in [16] an extensive analysis of the problem of source coding with side information, when the source is binary uniform, i.e., \( X \sim \text{Bernoulli}(0,5) \), and the correlation is given by a generic binary asymmetric channel:

\[
p(Y|X) = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix},
\]

\((a, b) \in [0, 1] \times [0, 1]\). The reconstructed source is the binary variable \( \hat{X} \in \hat{X} \), and the distortion metric considered is the Hamming distance: if \( (x, \hat{x}) \in X \times \hat{X} \) then \( d(x, \hat{x}) = 0 \) if \( x = \hat{x} \), and \( d(x, \hat{x}) = 1 \) if \( x \neq \hat{x} \).

In the predictive case our analysis led to the analytical formulation of the rate-distortion function for the problem (see [16], section II). For the general binary Wyner-Ziv problem, the nature of the underlying equations does not allow the formulation of a closed-form solution, but the proposed rate-distortion bound does not have an analytical expression.

In the Wyner-Ziv case, the general solution is given in terms of an auxiliary random variable, denoted by \( U \in \mathcal{U} \), satisfying the Markov conditions: \( U \leftrightarrow X \leftrightarrow Y \), and \( X \leftrightarrow (U, Y) \leftrightarrow \hat{X} \). It is known from [6] that the cardinality of this variable is bounded, namely \( \text{card } \mathcal{U} \leq \text{card } \mathcal{X} \). In order to describe the rate-distortion minimization problem in the Wyner-Ziv case using a mathematical apparatus, while keeping the complexity to a level that is tractable, we have made a simplifying assumption: we have considered the auxiliary random variable \( U \) to be binary. This actually implies that the bound we find might not be the rate-distortion function, but only an achievable bound. Nevertheless, at all times, a numerical computation of the rate-distortion function using the Blahut-Arimoto algorithm [17] as provided in [18] was used to confirm the tightness of our bound, i.e., the numerical algorithm presented in [16] describes the actual Wyner-Ziv rate-distortion function.

In this paper we complement the analysis of the binary Wyner-Ziv scenario from [16] by:

- extending the mathematical formulations in [16] to the case when the auxiliary variable satisfies the condition \( \text{card } \mathcal{U} = \text{card } \mathcal{X} + 1 = 3 \);
- showing that the minimization function in [16] reduces to a system of transcendental equations which do not have an analytical solution, but have a provable unique solution;
- showing the achievability of the rate-distortion function for both \( \text{card } \mathcal{U} = 2 \) and \( \text{card } \mathcal{U} = 3 \).

The rest of the paper is organized as follows: section II presents a description of the proposed bound when the auxiliary variable \( U \) is ternary; section III shows that the expression corresponding to the binary auxiliary variable admits an unique solution; section IV draws the conclusions of our work.

II. RATE-DISTORTION BOUND FOR TERNARY AUXILIARY VARIABLE \( U \)

In the case when the side information is available only at the decoder, the rate-distortion function was given by [6]:

\[
R_{WZ}(D) = \inf_{p(u|x)p(\hat{x}|u,y)} I(X; U|Y), \quad (2)
\]

where \( \hat{X} = f(U, Y) \) is a reconstruction function, such that \( E[\hat{d}(x, \hat{x})] \leq D \). Following the approach in [16], we assume that \( U \in \mathcal{U} \) is the outcome of a ternary channel with input \( X \) and the transition matrix given by:

\[
p(U|X) = \begin{bmatrix} m & n & (1 - m - n) \\ p & q & (1 - p - q) \end{bmatrix}, \quad (3)
\]

with \( 0 \leq m + n \leq 1 \) and \( 0 \leq p + q \leq 1 \). We express the formulas for the rate and distortion as functions of the crossover probabilities \((m, n)\) and \((p, q)\); we only point the essential steps in the derivation, but the interested reader is encouraged to follow the technicalities of the proof in [16].

From (1) and (10), we can write:

\[
p(U|Y) = \begin{bmatrix} (1-a)m+bp & (1-a)n+bp & (1-a)(1-m-n)+b(1-p-q) \\ (1-a+b) & (1-a+b) & (1-a+b) \end{bmatrix} \begin{bmatrix} am+(1-b)p & an+(1-b)q & a(1-m-n)+(1-b)(1-p-q) \\ (a+1-b) & (a+1-b) & (a+1-b) \end{bmatrix} \cdot (4)
\]

It follows that (2) can be written as:

\[
I(X; U|Y) = H(U|Y) - H(U|X) = \sum_y p(y)H(U|y) - \sum_x p(x)H(U|x), \quad (5)
\]

where \( H(U|*) \) is the conditional entropy function, which can be written using (10) and (4) as:

\[
H(U|*) = - \sum_u p(u|*) \log_2 p(u|*). \quad (6)
\]

The expression for the distortion can be written as shown in [16] to be:

\[
D = \sum_{u,y} \min( p(X = 0, y, u), p(X = 1, y, u) ). \quad (6)
\]

We can write \( p(X, Y, U) \) as follows:

\[
p(X = 0, Y, U) = \begin{bmatrix} (1-a)m & (1-a)n & (1-a)(1-m-n) \\ am & an & a(1-m-n) \end{bmatrix}, \quad (7)
\]

and

\[
p(X = 1, Y, U) = \begin{bmatrix} bp & bq & b(1-p-q) \\ (1-b)p & (1-b)q & (1-b)(1-p-q) \end{bmatrix}. \quad (8)
\]
Equations (7) and (8) can be used in (6) to write:

\[
D(m, n, p, q) = (\min((1-a)m, bp)) + \\
\min((1-a)n, bq)) + \\
\min((1-a)(1-m-n), b(1-p-q)) + \\
\min(a, (1-b)p)) + \\
\min(a, (1-b)q)) + \\
\min(a(1-m-n), (1-b)(1-p-q)).
\]

Equations (5) and (9) can be used to perform a full search of the space of the possible solutions \((m, n, p, q)\) of the following minimization problem:

- For each \(d \in [0, \frac{a+b}{2}]\) minimize \(R_{WZ}^*(m, n, p, q)\)
- Subject to: \(0 \leq D(m, n, p, q) \leq d \leq \frac{a+b}{2}\).

The example in Fig. 2 presents the result of the numerical analysis for \((a, b) = (0.1, 0.4)\) and \(|U| = 3\). It also shows the rate-distortion function as obtained in [16] and corroborated by the Blahut-Arimoto algorithm in [18]. One notices that the two bounds overlap. The main difference between the cases of the binary and ternary auxiliary variable is the following:

- For \(|U| = 3\), the space of the possible solutions has a higher dimensionality, so all the points on the rate-distortion function are directly achievable;
- For \(|U| = 2\), there is a time sharing region (convex envelope, i.e., the common tangent) that corresponds to the linear portion of the rate-distortion function.

In [16] we have conjectured that a binary auxiliary variable is sufficient to achieve the rate-distortion function for binary Wyner-Ziv coding. The conjecture is empirical and a direct mapping of the solutions in the ternary case to the solutions in the binary case seems improbable. Nevertheless, [19] and [20] have independently shown that, for lossless binary source coding with side information, the optimal size for the auxiliary variable is 2. A mathematical proof for our conjecture is foreseen as future work.

### III. Existence of an Unique Solution

In [16] we stated that the rate-distortion function for the binary Wyner-Ziv coding setup does not have a closed-form solution. In this section we will present the system of equations resulting from the optimization problem in [16]. In spite of their transcendental nature, we still prove that the system admits a solution which is unique.

Following the approach in [16], we assume that \(U \in \mathcal{U}\) in (2) is binary, i.e., it is the outcome of a binary channel with input \(X\) and the transition matrix given by:

\[
p(u|x) = \begin{cases} 
(1-p) & p \\
q & (1-q)
\end{cases}
\]

\((p, q) \in [0, 1] \times [0, 1]\). We express the formulas for the rate and distortion as functions of the crossover probabilities \((p, q)\); we skip the full derivation, which can be found in [16].

The rate function is:

\[
I(X;U|Y) \triangleq R_{WZ}^*(p, q) = H(U|Y) - H(U|X)
\]

\[
= \frac{1-a+b}{2} \cdot H\left(\frac{(1-a)p + b(1-q)}{1-a+b}\right)
\]

\[
+ \frac{1+a-b}{2} \cdot H\left(\frac{a(1-p) + (1-b)q}{1+a-b}\right)
\]

\[- \frac{1}{2} \cdot [H(p) + H(q)].
\]

We note that the rate-distortion formulation is reduced to the case \(0 < a + b < 1\) - see [16]. The reconstruction function is:

- If \(0 \leq d < \frac{a}{2(1-b)}\), we have only one reconstruction possibility, namely \(\hat{X} = U\)
- If \(\frac{a}{2(1-b)} \leq d \leq \frac{a+b}{2}\), we have two possible reconstructions, namely \(\hat{X} = U\) and \(\hat{X} = Y \cup U\).

The corresponding distortion functions are:

\[
D(p, q) = \begin{cases} 
D_1 = \frac{p+aq}{2}, & \text{if } \hat{X} = U \\
D_2 = \frac{(1-a)p+bq+aq}{2}, & \text{if } \hat{X} = Y \cup U
\end{cases}
\]

Given \(R_{WZ}^*(p, q)\) as in (11) and the expressions of the distortion function in (12), finding the bound can be formulated as a minimization problem with constraints:

- For each \(d \in [0, \frac{a+b}{2}]\) minimize \(R_{WZ}^*(p, q)\)
- Subject to: \(0 \leq D(p, q) \leq d \leq D_{max}\).
We will only consider the first case, i.e., \( D(p, q) = \frac{ap + bq}{2} \), as the other alternative follows identical reasonings, up to a constant term. We formulate the Lagrangian function associated with the minimization problem above:

\[
\mathcal{J}(p, q) = R_W^p(p, q) + \lambda(d - \frac{p + q}{2})
\]

and we set the partial derivatives with respect to \((p, q)\) to zero:

\[
\frac{\partial \mathcal{J}(p, q)}{\partial p} = 0 \quad \text{and} \quad \frac{\partial \mathcal{J}(p, q)}{\partial q} = 0
\]

This gives the following:

\[
\left\{
\begin{align*}
(1 - a) \log \frac{(1 - a)(1 - p) + bq}{(1 - a)p + b(1 - q)} - a \log \frac{a(1 - p) + (1 - b)q}{(1 - a)p + (1 - b)q} \\
- b \log \frac{(1 - a)(1 - p) + bq}{(1 - a)p + b(1 - q)} + (1 - b) \log \frac{ap + (1 - b)(1 - q)}{a(1 - p) + (1 - b)q} = \log 1 - p - \lambda \\
(1 - b) \log \frac{ap + (1 - b)(1 - q)}{a(1 - p) + (1 - b)q} = \log 1 - q - \lambda
\end{align*}
\right.
\]

where the logarithms are in the base 2. Removing \(\lambda\) from (13) and raising to the power 2 gives the following identity:

\[
\left(\frac{1 - a}{1 - a + b} + \frac{bq}{1 - a + b}\right)^{(1 - a + b) - a} - \frac{a}{(1 - a)p + b(1 - q)} \left(\frac{a(1 - p) + (1 - b)q}{1 - a + b}\right)^{1 - a + b - a} = \left(\frac{1 - p}{p} \cdot \frac{q}{1 - q}\right)^{1 - a + b - a}
\]

(14)

For a known distortion level \(d = D(p, q)\), there is an additional linear dependence between \(p\) and \(q\). In our case, by replacing \(q = 2d - p\), the above becomes a transcendental equation in \(p\), i.e., an identity involving both polynomial functions and functions that cannot be expressed as polynomials. It does not admit a closed-form solution. Nevertheless, we make the following claim:

**Proposition 1:** For every distortion level \(0 \leq d \leq \frac{a + b}{2}\), by imposing the distortion constraint in (12), the equation in (14) always admits a solution which is unique.

**Proof:** The proof is given in the appendix.

Essentially, the purpose of this analysis is twofold. First, it shows the nature of the equations that form our minimization problem, and justifies the use of a numerical algorithm to determine the solutions. Secondly, it shows that the problem is well posed and that, for correlation pairs \((a, b)\) in (1) such that \(0 < a + b < 1\), the solution to the minimization problem always exists and is unique. This implies that the proposed bound is achievable for any distortion \(d\).

**IV. Conclusions**

This paper complements our recent analysis of the problem of lossy binary source coding in the presence of correlated side information, presented in [16]. We have presented a descriptive analysis of the rate and distortion functions when the auxiliary variable \(U\) in (2) is ternary, and we have numerically confirmed that the bounds proposed for binary and ternary variables match. Moreover, in the case when the auxiliary variable \(U\) in (2) is binary, we have derived the transcendental equations that yield the optimal crossover probabilities required to reach the rate-distortion function. Even if the resulting system of equations is not analytically solvable, it is shown that it always admits a unique solution. In other words, the rate-distortion function is achievable for any target distortion.

**APPENDIX**

**Proof of Proposition 1**

Considering \(2d = p + q\), we make the following notations for the terms in (14):

\[
\begin{align*}
&f_1(p) = \frac{(1-a)(1-p)+b(2d-p)}{(1-a)p+b(1-2d+p)} \\
&f_2(p) = \frac{a(1-p)+(1-b)(2d-p)}{a(1-p)+(1-b)(2d-p)} \\
&g_1(p) = \frac{1-p}{p} \\
&g_2(p) = \frac{b}{1-2d+p}
\end{align*}
\]

For \(0 < a + b < 1\) as in [16] we derive the following:

\[
\left\{
\begin{align*}
\frac{\partial f_1(p)}{\partial p} &= -\frac{(1-a+b)^2}{[(1-a)p+b(1-2d+p)]} < 0 \\
\frac{\partial f_2(p)}{\partial p} &= -\frac{(1-a+b)^2}{[a(1-p)+(1-b)(2d-p)]} < 0
\end{align*}
\right.
\]

At the same time, we can write:

\[
\left\{
\begin{align*}
\frac{\partial g_1(p)}{\partial p} &= -\frac{1}{p} < 0 \\
\frac{\partial g_2(p)}{\partial p} &= -\frac{b}{1-2d+p} < 0
\end{align*}
\right.
\]

Given that \(\frac{\partial f(p)}{\partial x} = k f^{-1}(x) \frac{\partial f(x)}{\partial x}\) and considering the negativity of the terms in (15) and (16), we have that both terms of the equality in (14) are strictly decreasing functions in \(p\). Moreover, we observe that:

\[
\begin{align*}
\lim_{p \to 0} f_1(p) = C_1 \quad \text{and} \quad \lim_{p \to 1} f_1(p) = C_2 \\
\lim_{p \to 0} f_2(p) = C_3 \quad \text{and} \quad \lim_{p \to 1} f_2(p) = C_4 \\
\lim_{p \to 0} g_1(p) \cdot g_2(p) = +\infty \quad \text{and} \quad \lim_{p \to 1} g_1(p) \cdot g_2(p) = 0
\end{align*}
\]

(17)

where \(C_x\) are positive constants. Since \(g_1(p) \cdot g_2(p)\) is continuous, from (17) it follows that it is surjective on \([0, +\infty)\). As the left hand side term in (14) is also continuous, its values for \(p \in \{0, 1\}\) are in the open interval \((0, +\infty)\), and both terms of the equality are strictly decreasing, it follows that the two functions must intersect, but once and only once.
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