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Published in:
Mechanical Systems and Signal Processing

DOI:
10.1016/j.ymssp.2015.10.030

Publication date:
2016

Citation for published version (APA):

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Constrained maximum likelihood modal parameter identification applied to structural dynamics

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\textbf{A B S T R A C T}

A new modal parameter estimation method to directly establish modal models of structural dynamic systems satisfying two physically motivated constraints will be presented. The constraints imposed in the identified modal model are the reciprocity of the frequency response functions (FRFs) and the estimation of normal (real) modes. The motivation behind the first constraint (i.e. reciprocity) comes from the fact that modal analysis theory shows that the FRF matrix and therefore the residue matrices are symmetric for non-gyroscopic, non-circulatory, and passive mechanical systems. In other words, such types of systems are expected to obey Maxwell–Betti’s reciprocity principle. The second constraint (i.e. real mode shapes) is motivated by the fact that analytical models of structures are assumed to either be undamped or proportional damped. Therefore, normal (real) modes are needed for comparison with these analytical models. The work done in this paper is a further development of a recently introduced modal parameter identification method called ML-MM that enables us to establish modal model that satisfies such motivated constraints. The proposed constrained ML-MM method is applied to two real experimental datasets measured on fully trimmed cars. This type of data is still considered as a significant challenge in modal analysis. The results clearly demonstrate the applicability of the method to real structures with significant non-proportional damping and high modal densities.

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1. Introduction

Modal analysis is currently one of the key technologies used for analyzing the dynamic behaviour of complex structures such as cars, trucks, aircrafts, bridges, offshore platforms, and industrial machinery. During an experimental modal analysis (EMA) test, both the applied forces and vibration responses of the structure are measured when excited in one or more locations. After measuring the structure responses and the excitations forces, the modal parameters (i.e. resonance frequencies, damping ratios, mode shapes, and participation factors) are extracted from the measured frequency response functions (FRFs) by estimating an experimental parametric model of the structure under test using what is called system identification algorithms. Then, a modal model of the structure that essentially contains the same information as the original vibration data is constructed using those extracted modal parameters. The frequency-domain modal model expresses the behaviour of a linear time-invariant system as a linear combination of its different resonance modes as follows.
Active and semi-active vibration control

Forced response prediction

Structural modal correlation of damage detection

scaling factor). From the industrial point of view, the motivation to identify reciprocal models lies in the usage of such models for their structural dynamics research. Once the modal model is derived, a number of applications of modal analysis can be instigated. In the following, some of the applications of modal analysis are given:

- Damage detection
- Correlation of finite element model (FEM) and experimental results
- Structural modification
- Sensitivity analysis
- Forced response prediction
- Structure coupling/substructuring
- Active and semi-active vibration control

The modal model as it is represented by Eq. (1) takes into account only the linearity and the time-invariant constraints. However, for some applications (e.g. Correlation of FEM, structural modification, and substructuring [1-3]) a modal model that ensure a number of other constraints such as reciprocity (symmetry) of the residue matrices (i.e. \( R_L = R_L^T \)) and the proportional damping assumption (i.e. real mode shapes) is required. In this paper, a new modal parameter estimation method called ML-MM to directly establish modal models of structural dynamic systems satisfying the reciprocity and the real mode shapes constraints will be presented. The paper will be structured along the following lines: a review on the modal parameter estimation techniques that take into account those two constraints (i.e. reciprocity and real mode shapes) will be given in Section 2. The fast implementation of the ML-MM method will be presented in Section 3. In Section 4, the reciprocity and real mode shapes constraints will be applied to the ML-MM method. In Section 5, some validation results using some real industrial applications will be given to show the effectiveness of the constrained ML-MM method. Finally, some concluding remarks will be given in Section 6.

2. Constrained modal parameter estimation: a review

In practice, we often have some basic knowledge about the natural response of the structure under test based on physical insight. For instance, it is typically known whether the structure is stable. Then, it is desired that the identified model describes a stable system as well. Therefore, in many identification algorithms, the system poles are enforced to be either on the left-side of the s-plan in case of a continuous-time model has to be identified or inside the unit circle in case of identifying a discrete-time model [4-6].

Besides the system stability constraint, in modal analysis, when identifying structures a number of other fundamental properties are also desired and/or assumed. For instance, a certain MIMO linear time-invariant, non-lyrosropic, non-circular and passive mechanical system is expected to obey Maxwell–Betti’s reciprocity principle [7]. This implies that for such a system a partition of the frequency response functions matrix corresponding to collocated DOFs (i.e. degree of freedom of the structure where both the force and the response are measured) can be shown to be symmetric since the differential equation representing their behaviour is self-adjoint. Simply stated, a measurement with the excitation at point \( i \) and the response at point \( j \) is equal to the measurement with excitation at point \( j \) and the response at point \( i \). Mathematically speaking, in the modal model formulation, the residue matrix for each vibration mode and the upper and lower residuals terms used to compensate for the out-of-band modes have to be all symmetric. So, for a certain vibration mode the mode shape and corresponding modal participation vector are proportional to each other (i.e. identical up to a complex scaling factor). From the industrial point of view, the motivation to identify reciprocal models lies in the usage of such models in some important applications, e.g. structural modification prediction, substructuring [1,2]. For such practical applications, these experimental-driven models have to ensure a degree of physical feasibility since they will be used in a simulation environment (e.g. finite element modelling software). Therefore, having high quality reciprocal modal models is an important requirement from the theoretical and practical point of views.

Some authors have considered this constraint (i.e. reciprocity) in their identification algorithm. In [8], a constrained modal identification algorithm referred to as IDRM and uses the pole-residue model as a parameterization form is introduced. Amongst the applied constrains was the reciprocity property. In IDRM, the pole-residue model is optimized in an iterative way where the poles are iteratively updated, and the residues are calculated in a linear-least squares sense as a...
function of the updated poles and the measured FRFs at each iteration step without imposing reciprocity. Then, at each iteration step, the reciprocity constraint is applied by taking the symmetric part of the obtained residue matrix (i.e. $R_{ij} = (R_{ij} + R_{ji})/2$, [8]) and using a singular value decomposition to simultaneously enforce the minimality and the reciprocity. Minimality corresponds to the constraint on the rank-one property of the residue matrix found for single pole. Nothing has been mentioned by the author concerning imposing the reciprocity on the residual terms; however, it was mentioned in the conclusion of this article that issues linked to residual terms need to be further addressed. In [8], the author stated that the minimality is the most difficult constraint so that it will be the only constraint illustrated. Therefore, validation results related to the reciprocity constraint were not shown.

Methods to establish state-space models of structural dynamic systems satisfying physically motivated constraints, i.e. reciprocity, known static response, and displacement-velocity consistency criterion, are presented in [9]. The presented methods have been tested by means of a simulation example and an example with real test data as well. Both the considered simulation and real test data examples were representing a $2 \times 2$ MIMO system that exhibits reciprocity. In [9], it was shown that for SIMO, MISO and SISO systems, the global optimum is obtained already using a linear least-squares approach. For MIMO systems with enforced reciprocity however, the linear approach was shown to be incapable to attain the accurate models. In that case, the author recommended that a non-linear programming method has to be adopted. In [2], the possibility of modelling a machine tool setup using dynamic component synthesis with the objective to lay the foundation for chatter stability is investigated. The method introduced in that article is based on identifying subsystem state-space models that need to satisfy certain physically motivated constraints, e.g. reciprocity, and transforming these models to a coupling form. The authors of [2] addressed that enforcing the reciprocity on the identified subsystem state-space models deteriorates the identified model's overall correlation with the measured data, however the reciprocity constraint had to be imposed to ensure a degree of physical feasibility which is needed for successful component synthesis. A recent published method to symmetrize the residue matrix (i.e. the multiplication of the mode shape vector and the participation factors vector for mode $r \rightarrow R_r = \Psi_r L_r$) with the aim to have a reciprocal residue matrix is introduced in [10]. The method is simply done by first applying the well-known Polymax estimator [11,12] to the measured FRFs to get the poles and the participation factors for each identified mode. In a second step, the mode shapes and the residual terms are estimated by using the LSFD estimator [11,13] in a linear-least squares sense. The symmetry of the residue matrices is imposed iteratively in the LSFD step as following: each modal participation vector, $L^p_w$ with $p$ stands for the iteration number is replaced by the corresponding mode shape $\Psi^p_w$ and used together with the identified poles to estimate a new mode shapes set $\Psi^{p+1}_w$ until a kind of convergence on the residue matrix symmetry is reached. The effectiveness of that proposed method is measured by calculating and normalizing the skew part $(\Psi^p_w L^p_w - L^p_w \Psi^p_w)/(\Psi^p_w L^p_w + L^p_w \Psi^p_w)/2$ of a residue matrix $A_r = \Psi_r L_r \in C^{n \times n}$ with $B_r = A^T_r$, $i = 1, 2, \ldots, N_n$ and $j = 1, 2, \ldots, N_r$. This skew part should equal to zero in case of exact reciprocal residue matrix. The results of this approach showed that the skew part of the residue matrices was not exactly zero, which implies that the residue matrices are not exactly reciprocal. In addition, this approach does not impose the reciprocity on the residual terms (i.e. lower and upper residuals). Finally, convergence during the different iterations is not guaranteed.

Another constraint often needed in particular when comparing the experimental models with the analytical ones (e.g. FEM updating) is the estimation of real (normal) mode shapes instead of complex ones. The damping distribution in a structure determines whether the modes will be normal or complex. When a structure has very light or no damping it exhibits normal modes. If the damping is distributed in the same way as inertia and stiffness are distributed (proportional damping), we can also expect to find normal modes. Structures with very localized damping, such as car bodies with spot-welds and shock absorbers, have complex modes. Even in cases where the actual structure exhibits truly complex modes, the experimental identification of real modes, though by definition erroneous, is often still preferred in industrial practice for easiness of mode shape interpretation and comparison with the real (normal) FE modes. In the literature, it was shown that the real normal modes can be obtained by different ways. In [14], a general overview of these ways is given. These ways are mainly three. The first way is to measure directly the real modes using so called Phase Resonance Method which utilizes a harmonic excitation and an adjusted exciter force vector for each mode. The application of this method was found to be time consuming, and this led to develop another method which combined the phase resonance method with so called Phase Separation Techniques. The second way is to apply special Phase Separation Techniques to measured data. In this category, the methods are varied between time-domain and frequency-domain techniques. In the time-domain techniques, the product of the multiplication of the mass matrix inversion and the stiffness matrix is derived from the free decay vibrations, and then an eigenvalue problem is solved to have the real normal modes. This time domain technique was further developed by performing a principle component analysis. In the frequency-domain techniques, the method ISSP (Identification of Structural System Parameters) [15] and the FDFP (Frequency Domain Direct Parameter Identification) approach [16,17] enable to estimate real modes. In addition, it is proposed in [17] to estimate real (normal) mode shapes from a least squares approximation technique using Eq. (1) when the eigenvalues $\lambda_i$ and the participation factors $L_i$ are already known from applying e.g. the polyreference time domain method [18,19] or the polyreference least squares complex frequency-domain (pLSCF) estimator, industrially known as Polymax method [11,12]. The third way that can be used to identify real normal modes is to use a set of complex modes which already exists and has been previously identified with a Phase Separation Technique. This set of the complex eigenvectors together with their corresponding complex eigenvalues are then used to derive the real normal modes.

In [20], a method to identify the normal modes and associated non-proportional damping matrix was proposed. In this method, the scaled complex mode shapes are firstly identified using the IDR algorithm [8,21], in which the poles and residues are estimated by fitting a pole-residue model to the measured FRFs. Then, an approximation of the mass, damping and
stiffness matrices is calculated. In the last step, the real (normal) modes are determined as the eigenvectors of the conservative (based on mass and stiffness properties only) eigenvalue problem. This method introduced in [20] does not consider the effect of modes that are not within the selected frequency band (i.e. the residual effects), whereas, for instance as the author of [20] addressed, the component mode synthesis literature clearly indicates that the residual flexibility is quite essential for many problems. In [14], a method to derive the real modes from a pre-identified complex modes set is introduced. That method tried to solve the problem of the modal truncation by a reduction transformation. Modal truncation means that certain elements of the complex mode shape vector have to be selected for the mass and stiffness matrices calculation since the number of identified modes is always much less than the measured degree of freedoms (number of measured outputs). In the acoustic applications, more specifically the underwater acoustic waveguide, constraining the mode shapes to be real is essential as stated by [22]. In such application, the propagation physics constrain the mode shapes to be real. So, in [22], an approach to derive the real (normal) modes from a complex mode set using different phase rotation methods was described. The validation of this approach showed that the results were not satisfactory and did not match the authors’ expectations. The research on this method is still ongoing as it was stated by the author.

Recently, a modal parameter identification method called maximum likelihood modal model-based (ML-MM) has been introduced [23–28]. The basic implementation of the method is introduced in [24–27], while the computational speed of the method is significantly optimized in [23,28]. In the ML-MM method, the modal parameters are identified by fitting directly the modal model (see Eq. (1)) to the measured frequency-domain data (i.e. FRFs or positive power spectrum) in a maximum likelihood sense. The cost function to be minimized is the sum of the squared absolute value of the weighted-error between the model (expressed as "synthesized FRFs") and the FRF measurements. Weighted-error means that the variance of the measured FRFs is used as a weighting function to evaluate the quality of each frequency line in the cost function (i.e. high variance means small weight and hence less contribution in the cost function and vice versa). If the variance of the FRFs is not available, the ML-MM estimator becomes a non-linear least-squares estimator instead of maximum likelihood one. The ML-MM as it had been introduced does not apply any constraint to the modal model concerning the reciprocity and the real mode shapes estimation. In this paper, the further development of the ML-MM method that enables us to establish modal model that satisfies physical motivated constraints, i.e. the reciprocity of the FRFs and the real mode shapes estimation, will be presented, and validated using real measurements.

In the method we are introducing in this paper, there are some advantages over the previously existing methods. Firstly, using the modal model instead of the pole–residue model gives us the advantage of having automatically the minimality constraint on the residue matrix since the residue matrix will be the product of the multiplication of the mode shapes column vector and the participation factors row vector. So, applying a singular value decomposing (SVD) to the residue matrix to impose the minimality constraint as in [8] is avoided in the ML-MM method. When the minimality of the residue matrices is imposed by reducing them to rank-one matrices using the SVD technique, the quality of the optimized model is decreased. Secondly, the reciprocity constraint will not only be applied to the residue matrices as it was done in the literature but it will be also applied to the residuals terms (i.e. the upper and lower residual terms) which results in a full reciprocal modal model. However, the expansion of a reciprocal model of a rectangular system (i.e. more response DOFs than excitation DOFs are available) to a full square reciprocal system is beyond the objectives of this article. Thirdly, in the ML-MM method, the estimation of the real (normal) mode shapes does not require the derivation of an approximation for the mass and stiffness matrices as what had been done in some other methods in the literature. In the ML-MM method, the mode shapes will be directly imposed to be real-valued in the modal model (1), and then they will be iteratively optimized.

Fig. 1. Schematic representation of the fast ML-MM estimator.
using a non-linear optimization approach. Therefore, the estimation of real mode shapes through the ML-MM method will be done in a much more straightforward way.

3. ML-MM method: maximum likelihood modal model-based method

A comprehensive description of the ML-MM method is schematically represented in Fig. 1. Since ML-MM is an iterative method based on solving a non-linear optimization problem, initial values for the modal model parameters (i.e. Poles, Participation factors, mode shapes, lower and upper residuals) are needed to start the optimization process. So, the polyreference least-squares complex frequency domain (pLSCF) estimator \([11,12]\), industrially known as Polymax method, is applied to the FRFs to obtain initial estimates for the poles and the participation factors of the physical modes within the analysis band. Polymax has been selected because it gives a very clear stabilization chart in a fast way which makes it easy for the user to select the physical vibration modes within the analysis band. Then, initial values for the mode shapes and the lower and upper residuals are estimated in a complementary step using so called LSFD estimator \([13]\). The LSFD estimator fits the modal model (see Eq. (1)) to the measured FRFs in a linear least-squares sense. Since the poles and the participation factors are estimated previously using Polymax, the unknowns in Eq. (1), i.e. the mode shapes and the lower and upper residuals, can be easily determined in a linear least-squares sense. Once the initial values for the entire modal model parameters are obtained, the ML-MM solver starts to optimize that modal model (1) by minimizing the following cost function using a Gauss-Newton optimization assuming that correlation does not exist over either the different outputs or the frequency lines:

\[
K_{ML-MM}(\theta) = \sum_{o=1}^{N_o} \sum_{k=1}^{N_f} E_o(\theta, \omega_k) E_o^H(\theta, \omega_k)
\]

(2)

with \(N_f\) the number of the frequency lines, \(N_o\) the number of the measured responses (outputs), \(\theta\) the complex conjugate transpose of a matrix, \(\omega_k\) the circular frequency, \(\theta\) the model parameters vector, and \(E_o(\theta, \omega_k)\) the (weighted) error equation corresponds to the \(o^{th}\) output degree of freedom (DOF) given as follows:

\[
E_o(\theta, \omega_k) = \begin{bmatrix}
\frac{H_{oo}(\omega_k) - \hat{H}_{oo}(\theta, \omega_k)}{\text{var}(H_{oo}(\omega_k))} & \cdots & \frac{H_{om}(\omega_k) - \hat{H}_{om}(\theta, \omega_k)}{\text{var}(H_{om}(\omega_k))}
\end{bmatrix} \in C^{1 \times N_i}
\]

(3)

where \(H_{oo}(\omega_k) = [H_{o1}(\omega_k) \ H_{o2}(\omega_k) \ \cdots \ H_{oN_i}(\omega_k)] \in C^{1 \times N_i}\) is the \(o^{th}\) row of the measured FRFs matrix with \(N_o\) the number of inputs, \(\text{var}(H_{oo}(\omega_k)) \in R^{1 \times N_i}\) is the variance of the \(o^{th}\) row of the measured FRFs matrix, and \(\hat{H}_{oo}(\theta, \omega_k) = [\hat{H}_{o1}(\theta, \omega_k) \ \hat{H}_{o2}(\theta, \omega_k) \ \cdots \ \hat{H}_{oN_i}(\theta, \omega_k)] \in C^{1 \times N_i}\) is the \(o^{th}\) row of the estimated (synthesized) FRFs. The error equation \(E_o(\theta, \omega_k)\) can be written differently in a logarithmic scale which leads to the logarithmic implementation of the ML-MM solver. The logarithmic error equation is written as follows:

\[
E_{\text{log}}(\theta, \omega_k) = \begin{bmatrix}
\log \left( \frac{\hat{H}_{o1}(\theta, \omega_k)}{H_{o1}(\omega_k)} \right) \\
\cdots \\
\log \left( \frac{\hat{H}_{oN_i}(\theta, \omega_k)}{H_{oN_i}(\omega_k)} \right)
\end{bmatrix} \in C^{1 \times N_i}
\]

(4)

The logarithmic cost function \([29,30]\) is found to be more robust to the noise assumptions made as well as to the outliers, and can handle measurements with a large dynamic range as it was stated in \([29]\). In Eqs. (3) and (4), the estimated FRFs \(\hat{H}_{oo}(\theta, \omega_k)\) are represented by the modal model formulation (1). The optimization of the modal model parameters is done in two stages. At each iteration, the ML-MM solver considers the poles (\(\lambda_i\)) and the participation factors (\(L_i\)) as the parameters to be updated using the Gauss–Newton optimization, while the mode shapes (\(\Psi_i\)) together with the lower and upper residuals (\(LR\) and \(UR\)) are estimated as implicit functions of the poles and the participation factors. The Gauss–Newton iterations are given by

\[
\begin{bmatrix}
f_p^H f_p \\
\end{bmatrix} \delta \theta_p = -J_p^H E_p
\]

(5)

\[
\theta_{p+1} = \theta_p + \delta \theta_p
\]

(6)

with \(J_p = \frac{\partial E_p}{\partial \theta_p} \in C^{N_o N_i \times 2N_o N_i(N_i + 1)}\) the Jacobian matrix containing the derivatives of the equation error with respect to the real and imaginary parts of the participation factors and the poles for all the modes at iteration \(p\), \(E_p \in C^{N_o N_i \times 1}\) the errors between the measured FRFs and the modal model (1) for all the FRFs and at all the frequency lines, and \(\delta \theta_p = [\delta \theta_p \ \delta \theta_p]^T \in R^{2N_o (N_i+1) \times 1}\) the perturbations on the participation factors and the poles. The Jacobian matrix \(J_p\) is given as follows:

\[
J_p = \begin{bmatrix}
\Gamma_{11} & 0 & \cdots & 0 & \Phi_{11}^2 \\
0 & \Gamma_{12} & \cdots & 0 & \Phi_{12}^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \Gamma_{N_i} & \Phi_{N_i}^2
\end{bmatrix} \in C^{N_o N_i \times 2N_o N_i(N_i + 1)}
\]

(7)
with \( \Gamma_{Li} \in \mathbb{C}^{NfN_i \times 2Nm} \) the derivatives of the equation error with respect to the real and imaginary parts of the participation factors of all the modes for the input \( i \) where \( i = 1, 2, \ldots, N_i \) and \( \Phi_{Li}^H \in \mathbb{C}^{NfN_i \times 2Nm} \) the derivatives of the equation error with respect to the real and imaginary parts of the poles of all the modes for input \( i \). Taking into account the structure of that Jacobian matrix, the normal Eq. (5) can be written as follows:

\[
\begin{bmatrix}
R_{i1} & 0 & \ldots & 0 & S_{i1}^L \\
0 & R_{i2} & \ldots & 0 & S_{i2}^L \\
0 & 0 & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
S_{1i}^L & S_{2i}^L & \ldots & S_{Ni}^L & \sum_{i=1}^{N_i} T_{Li}
\end{bmatrix}
\delta \theta_p = -
\begin{bmatrix}
F_1 \\
F_2 \\
\vdots \\
F_{Ni}
\end{bmatrix}
\]  

(8)

with \( R_{Li} = \mathcal{R}(\Gamma_{Li}^H) \in \mathbb{R}^{2Nm \times 2Nf} \), \( S_{Li}^L = \mathcal{R}(\Gamma_{Li}^H \Phi_{Li}^H) \in \mathbb{R}^{2Nf \times 2Nm} \), \( T_{Li} = \mathcal{R}(\Phi_{Li}^H \Phi_{Li}^H) \in \mathbb{R}^{2Nf \times 2Nf} \), \( F_i = \mathcal{R}(\Gamma_{Li}^H \text{vec}(E_i)) \), \( V_i = \mathcal{R}(\Phi_{Li}^{H'} \text{vec}(E_i)) \) where vec(\( E_i \)) \( \in \mathbb{C}^{N_fN_i \times 1} \) the error vector corresponds to input location \( i \) and all the outputs at all the frequency lines. Using the matrix inversion lemma [31] together with some elimination and substituting procedures, the perturbations on the real and imaginary parts of the poles \( \delta_{\lambda_p} \) and the participation factors \( \delta_{\alpha_p} \) at iteration \( p \) are given by

\[
\delta_{\lambda_p} = \left[ \sum_{i=1}^{N_i} T_{Li} - S_{Li}^L \right]^{-1} \left[ \sum_{i=1}^{N_i} S_{Li}^L (R_{Li})^{-1} F_i - V_i \right] \in \mathbb{R}^{2Nf \times 1}
\]

(9)

\[
\delta_{\alpha_p} = -(R_{Li})^{-1} \left[ F_i + S_{Li}^L \delta_{\lambda_p} \right] \in \mathbb{R}^{2Nf \times 1}
\]

(10)

Once the perturbations on the parameters (i.e. \( \delta_{\lambda_p} \) and \( \delta_{\alpha_p} \)) are calculated, the poles and the participation factors are updated to be used for a new iteration as follows:

\[
\lambda_{p+1} = \lambda_p + \delta_{\lambda_p}
\]

\[
L_{p+1} = L_p + \delta_{\alpha_p}
\]

(11)

The initial parameters estimates \( \theta_0 \) (starting values) to construct the normal equations in the first iteration are estimated by Polymax estimator. In fact, the ML-MM solver optimizes the results obtained by the least-squares procedures. From practical studies, a serious improvement of the accuracy can be achieved using Gauss–Newton optimization procedure (see Eqs. (5) and (6)), while the use of a Levenberg–Marquardt loop [32] forces the algorithm to converge by solving the following equation (i.e. Eq. (12)) instead of Eq. (5):

\[
\begin{bmatrix}
J_{p,lp}^H + \alpha_{lm} \text{diag}(J_{p,lp}^H) \\
0
\end{bmatrix} \delta \theta_p = -J_{p,lp}^H E_p
\]

(12)

Increasing the parameter \( \alpha_{lm} \) forces the cost function to decrease, but decreases the convergence speed. Therefore, this factor \( \alpha_{lm} \) is adapted in every iteration depending on the evolution of the cost function. In each iteration, after updating the poles and the participation factors the mode shapes and the upper and lower residuals are calculated in a linear least-squares formulation using the updated poles, the updated participation factors, and the measured FRFs. This can be easily illustrated by writing the modal model (1) in a matrix form for all the values of the frequency axis as follows:

\[
\begin{bmatrix}
\hat{Y} \\
[LR] \\
[UR]
\end{bmatrix} = [A(L, \lambda, j\omega)]
\]

(13)

with \( [Y] \in \mathbb{R}^{Nf \times 2Nf} \) a matrix containing the real and imaginary parts of the mode shapes of all modes \( (\Psi_1, \Psi_2, \ldots, \Psi_{N_m}) \), \( [LR] \in \mathbb{R}^{Nf \times N_f} \) the lower residuals matrix, \( [UR] \in \mathbb{R}^{Nf \times N_f} \) the upper residuals matrix, \( [A(L, \lambda, j\omega)] \in \mathbb{R}^{(2Nf+2N_i) \times 2Nf} \) a matrix which is an explicit function of the poles, participation factors, and Laplace variable \( j\omega \), and \( [\hat{H}] \in \mathbb{R}^{N_f \times 2Nf} \) (defined in Appendix C) a matrix containing the real and imaginary parts of the frequency response functions (FRFs). By replacing \( \hat{H} \) with the measured ones \( H \) and using the updated poles and participation factors (see Eq. (11)) in calculating the entries of the matrix \( A(L, \lambda, j\omega) \), the mode shapes and the residual matrices can be determined in a linear least-squares sense as follows:

\[
\begin{bmatrix}
[Y] \\
[LR] \\
[UR]
\end{bmatrix} = [H][A(L, \lambda, j\omega)]'
\]

(14)

Once the mode shapes and the residuals matrices are calculated, all these optimized modal parameters (i.e. \( \lambda, L, \Psi, LR, \) and \( UR \)) are used to start a new iteration in the optimization loop till either the cost function reaches a convergence defined by a relative error between two consecutive calculated cost functions or the maximum number of iterations is reached. In case the FRFs variance has been used as a weighting function (see Eq. (3)), the confidence bounds on the perturbed parameters (i.e. \( L, \lambda \)) can be obtained from the inversion of so called Fisher information matrix [32,33] as follows:

\[
cov(L, \lambda) = \left[ \mathcal{R}(\hat{J}^H) \right]^{-1}
\]

(15)
4. Constrained ML-MM method

In this section the implementation of the ML-MM solver described in Section 3 will be adapted in the way which takes into account two physically motivated constraints during the modal parameter estimation process. Those two constraints are: the FRFs reciprocity and the real mode shapes estimation.

4.1. Reciprocity

Most often it is assumed that the system under test obeys Maxwell–Betti’s reciprocity principle: a measurement with the excitation at point \( i \) and the response at point \( j \) is equal to the measurement with excitation at point \( j \) and the response at point \( i \). A reciprocal frequency response function (FRF) matrix requires symmetric residues and symmetric residual matrices. If MIMO measurements are available, this condition can be checked on the FRFs: \( H_{ij}(\omega_k) = H_{ji}(\omega_k) \). On the level of the modal model, evaluating Eq. (1) for point \( i \) and \( j \) shows that the reciprocity principle for a specific mode \( r \) and the upper and lower residual terms yields:

\[
\begin{align*}
\Psi_{iL}L_i &= \Psi_{jL}L_i \\
LR_{ij} &= LR_{ji} \\
UR_{ij} &= UR_{ji}
\end{align*}
\]

Hence, to identify a reciprocal modal model with the ML-MM method the residual matrices have to be symmetric and the participation factors have to be identical to the mode shape coefficients up to scaling factor at the input stations. Therefore, in the case the reciprocity condition is applied and the number of the measured outputs equals the number of the measured inputs, the modal model represented by Eq. (1) can be written as follows (assuming displacement FRFs):

\[
\dot{H}(\theta, \omega_k) = \left( \sum_{i=1}^{N_o} Q_{ri} \Psi_i \Psi_i^T \right) f_{i} + \frac{1}{\lambda_{i} - \lambda_{r}} \left[ LR_{i} \right]_{S} + \left[ UR_{i} \right]_{S}
\]

(19)

with \( Q_{r} \in \mathbb{C} \) the scaling factor for the \( r \)th mode, \( \left[ LR_{i} \right]_{S} \in \mathbb{S}^{N_o} \) (\( \mathbb{S} \) : square matrix) and \( \left[ UR_{i} \right]_{S} \in \mathbb{S}^{N_o} \) the symmetric lower and upper residual matrices. But, in modal testing it is often that the number of the measured outputs is much higher than the number of the inputs (i.e. \( N_o > N_i \)), hence the frequency response matrix is not square anymore. In this case, the length of the right and left eigenvectors of the residue matrices (i.e. participation factors and mode shapes vectors) is different, and the symmetry property of the residues and residual matrices is only valid between a group of the off-diagonal elements of the matrix and not for all the elements. These reciprocal (symmetric) elements belong to a partition of those matrices which corresponds to collocated DOFs (i.e. degree of freedom of the structure where both the force and the response are measured). For the general case where the number of the outputs and inputs differs (\( N_o \neq N_i \)), Eq. (19) can be written in a more general form as follows:

\[
\dot{H}(\theta, \omega_k) = \left( \sum_{i=1}^{N_o} Q_{ri} \phi_i \phi_i^T \right) f_{i} + \frac{1}{\lambda_{i} - \lambda_{r}} \left[ LR_{i} \right]_{rec} + \left[ UR_{i} \right]_{rec}
\]

(20)

with \( \phi_i \in \mathbb{C}^{N_o \times 1} \) the mode shapes vector in which the shape coefficients correspond to the excitation points indexes (i.e. the excitation locations) are considered to be identical to the corresponding participation factors up to scaling factor \( Q_{r} \) and \( \left[ LR_{i} \right]_{rec} \in \mathbb{S}^{N_o \times N_o} \) and \( \left[ UR_{i} \right]_{rec} \in \mathbb{S}^{N_o \times N_o} \) the reciprocal lower and upper residual matrices. So, assume a certain structure that has DOFs as 1, 2, 3, ……, \( N_o \) where \( N_o \) is the number of the measured outputs. That structure is excited on the first \( N_i \) DOFs where \( N_i \) is the number of the measured inputs. The participation factors vector \( L_r \) and the mode shapes vector \( \phi_r \) for a certain mode \( r \) are given as follows:

\[
\begin{align*}
L_r &= [ l_1 \ l_2 \ \ldots \ l_{N_i} ]_T \in \mathbb{C}^{1 \times N_i} \\
\phi_r &= [ \psi_{N_i+1} \ \psi_{N_i+2} \ \ldots \ \psi_{N_o} ]_T \in \mathbb{C}^{N_o \times 1}
\end{align*}
\]

(21)

with \( l \) and \( \psi \) are the entries of the \( r \)th participation factors vector \( L_r \) and the \( r \)th mode shapes vector \( \phi_r \) respectively. In this equation, it can be clearly seen that the mode shapes coefficients at the locations of the driving points are taken to be equal to the participation factors through a scaling factor. Having \( L_r \) and \( \phi_r \) as they are described in the equation above, the product of the \( Q_r \phi_r L_r \) multiplication yields an exact reciprocal residue matrix with a dimension \( N_o \times N_i \). For the lower and
upper residual matrices, the reciprocity constraint is achieved by imposing that \((LR_{ij} = LR_{ji})\) and \((UR_{ij} = UR_{ji})\) with \(i\) goes for the rows that correspond to the collocated DOFs and \(j\) corresponding to the columns. Therefore, using the ML-MM method, the reciprocity constraint will be taken into account during the modal parameter identification by minimizing the error between the modal model proposed in Eq. (20) and the measured FRFs. The fitting will be done either in a maximum likelihood sense or in a non-linear least-squares sense depending on the availability of the variance of the noise on the measured FRFs. To adapt the ML-MM method introduced in Section 3 to this constraint, the bulk of the modifications had to be implemented in the part of the method where the mode shapes and the residual matrices are calculated as implicit functions of the poles and participation factors (see Eqs. (13) and (14)). The procedure of applying the reciprocity constraint on the ML-MM method will be done as following:

(1) The Polymax (pLSCF) estimator is applied to the measured FRFs with the aim to generate initial values for the poles \(\lambda_r\) and participation factors \(L_r\) for each mode.

(2) Taking the reciprocity constraint into account, the modal model proposed in Eq. (20) with \(\lambda_r\) and \(L_r\) known from step 1 is fitted to the measured FRFs in a linear least-squares sense with the aim to estimate initial values for the unknown scaling factors \(Q_r\), mode shape \(\phi_r\), the lower residual matrix \([LR]_{\text{rec}}\), and the upper residual matrix \([UR]_{\text{rec}}\). This step is done as follows:

a) The measured FRFs matrix and the unknown residual matrices (i.e. \([LR]\) and \([UR]\)) will be divided into two partitions: reciprocal and non-reciprocal partitions. The reciprocal partition is a square matrix with a dimension \(N_l \times N_l\) which corresponds to collocated DOFs (i.e. driving points) and it will be denoted as \([L]_{\text{rec}}\). This reciprocal partition of the measured FRFs matrix will be utilized to estimate the scaling factors \(Q_r\) (with \(r = 1, \ldots, N_m\)) and the reciprocal partition of the residual matrices. This will be done by solving the following equation in a linear least-squares sense for those unknowns:

\[
[H] = \begin{bmatrix}
A_Q & A_{UR}
\end{bmatrix} \begin{bmatrix}
Q \\
LR_{\text{sub}} \\
UR_{\text{sub}}
\end{bmatrix},
\]

with \([H] = [\Re(\text{vec}([H(\omega_k)]_{\text{rec}})); \Im(\text{vec}([H(\omega_k)]_{\text{rec}}))] \in \mathbb{R}^{2N_l^2N_l \times 1}\) (22)

where \([H(\omega_k)]_{\text{rec}}\) is the reciprocal square partition of the FRFs matrix, \(A_Q \in \mathbb{R}^{2N_l^2N_l \times 2N_m}\) (defined in Appendix A) is function of the poles and participation factors, \(A_{UR} \in \mathbb{R}^{2N_l^2N_l \times N_l(N_l+1)}\) (defined in Appendix A) counts for the residual terms, \(Q \in \mathbb{R}^{2N_m \times 1}\) is containing the real and imaginary parts of the scaling factor for each mode, \(LR_{\text{sub}} \in \mathbb{R}^{N_l(N_l+1) \times N_l}\) and \(UR_{\text{sub}} \in \mathbb{R}^{N_l(N_l+1) \times N_l}\) are containing a subset from the elements of the lower and upper residual matrices, the diagonal elements plus half of the off-diagonal elements. In case the lower and upper residual terms are not taken into the identified modal model, the matrix \(A_{UR}\) does not exist. In case that either only the lower residual or only the upper residual is taken into the model, the number of columns in matrix \(A_{UR}\) is halved (i.e. \(A_{UR} \in \mathbb{R}^{2N_l^2N_l \times (N_l+1)/2}\)). Once the subset of elements of the residual matrices is estimated (i.e. \(LR_{\text{sub}}\) and \(UR_{\text{sub}}\)), the full elements of the reciprocal residual matrices are found using a transformation matrix \(T\) as follows:

\[
LR_{\text{full}} = TLR_{\text{sub}} \in \mathbb{R}^{N_l \times 1}
\]

\[
UR_{\text{full}} = TUR_{\text{sub}} \in \mathbb{R}^{N_l \times 1}
\]

The transformation matrix \(T\) is described in Appendix A.

b) Then, the non-reciprocal partitions of the FRFs matrix \([H]_{\text{nonRec}}\) together with the known poles, participation factors, and the estimated scaling factors \(Q\) calculated from step (2.a) are used to estimate the non-reciprocal partition of the residual matrices together with the mode shape coefficients that do not correspond to the driving points (i.e. the excitation locations). This will be done by solving an equation similar to Eq. (14), as follows:

\[
[H'] = [LR'; UR'] = H_{\text{nonRec}} \begin{bmatrix} A(Q, L, \lambda, j\omega) \end{bmatrix}^T
\]

with \([H'] \in \mathbb{R}^{N_l \times 2N_m}\) a matrix containing the real and imaginary parts of the mode shape elements of all modes \((\Psi_1, \Psi_2, \ldots, \Psi_{N_m})\), \([H]_{\text{nonRec}} \in \mathbb{R}^{N_l \times 2N_m}\), and \([A(Q, L, \lambda, j\omega)] \in \mathbb{R}^{2N_m \times 2N_m}\). In Eq. (24), one can notice that matrix \(A\) is now a function of one more parameter which is the scaling factor \(Q\). Now, the initial values for all the parameters of the modal model (20) have been identified taking into account the reciprocity constraint.

(3) The poles \(\lambda_r\) and the participation factors \(L_r\) are then updated using the optimization technique described in Section 3. The formulation of the modal model described by Eq. (20), which is needed when imposing the reciprocity constraint, will be used to calculate the error \(E_p\) and the entries of the Jacobian matrix \(J_p\). Then, \(E_p\) and \(J_p\) will be used in Eq. (12) to calculate the perturbations on \(\lambda_r\) and \(L_r\) \(\delta\theta_p = \begin{bmatrix} \delta\lambda_r \\ \delta L_r \end{bmatrix}^T\).
(4) Once $\lambda_r$ and $L_r$ are updated, step 2 is repeated to calculate new scaling factors, new shape coefficients, and new residual matrices. So, steps 2 and 3 are iteratively done until the convergence criteria of the cost function are reached.

4.2. Real (normal) mode constraint

Estimation of real (normal) mode shapes requires that the structure under test has a proportional damping which is a quite hypothetical form of damping. The main reason for the introduction of the proportionally damped systems assumption is that the numerical complexity of the calculations with this assumption is lower than for the general viscous damping. Systems with proportional damping form a compromise between the undamped system models from finite element model analysis and the generally viscously damped system models from experimental modal analysis. The hypothesis of proportional damping of a given mode corresponds to a purely imaginary residue matrix $[8,13]$. This corresponds to $\Re(\Psi,L_r)=0$ in Eq. (1). The modal model, which will be optimized using the ML-MM method in case the real mode constraint is taken into account, is given by Eq. (25). One can see from this equation that it corresponds to purely imaginary residue matrices as it was stated in $[8,13]$ for the case of proportional damping assumption.

$$
\begin{align*}
\tilde{H}(\theta,\omega) &= \left( \sum_{j=1}^{N_f} \frac{J_p L_r}{j\omega - \lambda_r} + \frac{-J_p L_r}{j\omega - \bar{\lambda}_r} \right) + \frac{[LR]}{(j\omega)^y} + [UR] \\
\end{align*}
$$

In Eq. (25), $\Psi_r \in \mathbb{R}^{N_d \times 1}$ is the real-valued mode shapes coefficients, $L_r \in \mathbb{R}^{1 \times N_f}$ is the real-valued participation factors, and the other parameters remain the same as they are defined in Eq. (1). Good starting values for the real-valued participation factors are obtained by taking the magnitude of the complex-valued ones obtained from the initial values generator (i.e. Polymax estimator/see Fig. 1) multiplied with the sign of their real part. So, the ML-MM method will estimate a modal model with real-valued mode shapes by fitting the modal model (25) to the measured FRFs iteratively in a maximum likelihood sense using Levenberg-Marquardt optimization technique. The procedure can be decomposed into the following steps:

1. Initial values for the complex-valued participation factors, $L_r \in \mathbb{C}^{1 \times N_f}$, and the poles $\lambda_r \in \mathbb{C}$ are obtained by applying the Polymax (pLSCF) estimator to the measured FRFs.
2. The complex-valued participation factors are transformed to real-valued ones by taking the magnitude of the complex value multiplied with the sign of the real part, $L_r \in \mathbb{R}^{1 \times N_f}$.
3. Initial values for the real-valued mode shapes together with the residual matrices are obtained from Eq. (14). Since the mode shapes and the participation factors are taken now as real-valued parameters, the number of columns in matrix $[\Psi]$ and the number of rows of matrix $[A(L,\lambda,\omega)]$ will be halved. $[\Psi] \in \mathbb{R}^{N_d \times N_d}$ and $[A(L,\lambda,\omega)] \in \mathbb{R}^{(N_d + 2N_f) \times 2N_d}$. 
4. The modal model (25) is used to calculate the error $E_p$ and the entries of the Jacobian matrix $J_p$ that will be used in Eq. (12) to calculate the perturbations on $\lambda_r$ and the real-valued $L_r$ (i.e. $\delta \theta_p = \begin{bmatrix} \delta \lambda_r \\ \delta \Psi_r \\ \delta L_r \\ \delta A \\ \delta R \\ \delta \Psi_r \\ \delta L_r \\ \delta A \\ \delta R \end{bmatrix}$) with the submatrices $\Gamma_{\Psi} \in \mathbb{R}^{2N_d \times N_d}$, $\Gamma_{L} \in \mathbb{R}^{N_d \times N_d}$, $\Gamma_{\Psi} \in \mathbb{R}^{N_d \times 2N_d}$, $\Gamma_{L} \in \mathbb{R}^{N_d \times N_d}$, $\delta_{\Psi} \in \mathbb{R}^{N_d \times 1}$, and $\delta_{\Psi} \in \mathbb{R}^{2N_d \times 1}$.
5. The updated poles and real-valued participation factors are then used to calculate updated real-valued mode shapes and residuals matrices using Eq. (14).
6. Steps 4 and 5 are iteratively done until the convergence criteria of the cost function (2) are reached.

In case both reciprocity and real modes constraints are needed, the ML-MM method will identify a modal model represented by Eq. (25) in the way described in Section 4.1.

5. Validation and discussion

To highlight the effectiveness of the proposed constrained ML-MM method for the identification of a constrained modal model, applications to real measurement cases will be shown in this section. Since the main objective of the users who apply such constraints is to obtain models that accurately represent the system under test, the main criterion that is going to be utilized as a validation tool is the quality of the fit between the obtained modal model and the measured data. In automotive engineering, experimental modal analysis (EMA) is considered as a “commodity” tool and accurate models are needed for modelling and finite element updating. A typical example of a challenging modal analysis application is the structural analysis of a trimmed car body. In this section, the proposed constrained ML-MM estimator will be validated by means of two data sets measured from two different fully trimmed cars.

5.1. First fully trimmed car example

This data set consists of 616 FRFs measured for a fully trimmed Porsche car. The accelerations of the fully equipped car were measured at 154 locations, while four shakers were simultaneously exciting the structure. This gives a total of 616 FRFs. At the shaker locations, accelerometers were also installed to measure the accelerations. Therefore, the data set has
four driving points; hence a $4 \times 4$ partition of the full FRFs matrix is expected to be symmetric due to the fact that the FRFs of the collocated DOFs should be reciprocal. In the following subsections, the validation results of the ML-MM method applied with the reciprocity and real mode shapes constraint will be shown.

5.1.1. Reciprocity constraint

As it was mentioned in Section 3, to start the ML-MM estimator initial values for all the modal model parameters (see Eq. (1)) will be obtained by applying Polymax estimator followed by the LSFD estimator to the measured FRFs in a first step. Fig. 2 shows the stabilization chart constructed using the Polymax estimator. The stabilization chart shows that there are around 18 physical modes in the analysis band. Those 18 modes have been selected from the stabilization chart at the maximum model order. The mode shapes together with the lower and upper residuals are calculated in a linear least-squares sense using the LSFD estimator which calculates them as implicit functions of the poles and participation factors. Starting from these initial values for the modal model parameters, the ML-MM estimator was then utilized to optimize the modal model (20) in order to minimize the cost function (2). To check the effect of the reciprocity constraint on the quality of the obtained modal model, the LSFD and ML-MM approaches are used both with and without applying the reciprocity constraint. Without applying the reciprocity constraint means that the modal model (1) will be optimized instead of (20). In case reciprocity constraint is applied, the modal model represented by equation (20) is identified through the steps listed in Section 4.1.

Fig. 3 shows the minimization of the ML-MM cost function (2) during the different iterations for both with and without the reciprocity constraint cases. The number of iterations has been selected to be 100 iterations, because it was noted that the error in case of the unconstrained ML-MM was noticeably decreasing with increasing the iterations. For this data set (616 FRFs), the calculation time taken by the ML-MM to achieve those 100 iterations was about 137 s. One can see that the initial error in case of constrained ML-MM is dramatically higher than the initial error in case of unconstrained ML-MM. This is can be also noted from Fig. 4 in which a synthesized FRF generated by the LSFD estimator is compared to the measured one. One can notice from that figure that applying the reciprocity constraint in the LSFD estimator (i.e. linear least-squares solution) leads to erroneous synthesized FRFs in comparison to the unconstrained case. However, the ML-MM estimator is

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**Fig. 2.** The 1st fully trimmed-car example: Stabilization chart constructed by LMS Polymax.

**Fig. 3.** The 1st fully trimmed-car example: Decreasing of the error during the reciprocity constrained and unconstrained ML-MM.
capable of reducing the error significantly. The high initial error in case of applying the reciprocity constraint shows that the linear least-squares solution that is being utilized in the LSFD estimator is not sufficient to obtain an accurate reciprocal modal model. This shows the need for more advanced estimators that could maintain such physically motivated constraint without losing the quality of the fit between the identified models and the measured data. This was actually the main motivation behind introducing the constrained ML-MM estimator that we are presenting in this article. For some selected FRFs, the quality of the fit between the synthesized and the corresponding measured ones is shown in Fig. 5 for FRFs corresponding to driving point measurement where the reciprocity is expected, and in Fig. 6 for FRFs corresponding to non-driving point locations. One can see in Fig. 5 that the measured FRFs are reasonably reciprocal. However, there are still some differences which could be related to some experimental errors e.g. alignment of collocated sensors, errors in sensors calibration values. The results shown in Fig. 5 show that the proposed constrained ML-MM estimator leads to FRFs matrix that is exactly symmetric where it can be seen that the FRFs that are expected to be reciprocal are exactly the same. Moreover, comparing the initial fit obtained from the constrained LSFD step to the final fit we got after the iterations of the constrained ML-MM estimator shows that the quality of the obtained reciprocal modal model is improved. The same remark can be drawn from the results shown in Fig. 6 as well. By comparing the initial fit (i.e. LSFD synthesized) in both Figs. 5 and 6, one can see that the degradation of the quality of the fit due to the application of the reciprocity constraint is less severe for the FRFs that correspond to collocated DOFs (driving points) in comparison with the others FRFs. As a global indication of the quality of the obtained modal model, Table 1 represents the mean fitting error and correlation between the measured and synthesized FRFs. The mean fitting error and correlation are calculated using the equation presented in appendix B. The values shown in that table show the robustness of the proposed constrained ML-MM estimator to deliver an accurate reciprocal modal model. Moreover, it can be seen from Table 1 that the constrained ML-MM solution is even better than the unconstrained LSFD method.

5.1.2. Real Mode shapes estimation

In this subsection, the ML-MM solver constrained for estimating real mode shapes will be validated. For the same data set, the constrained ML-MM solver will be utilized to identify the modal model (25), which incorporates real mode shapes, using the approach described in Section 4.2. Both the logarithmic (see Eq. (4)) and linear (see Eq. (3)) implementations of the ML-MM solver have been applied. It is found that the logarithmic one gives better results, so its result will be shown. 20 iterations are used for the iterative ML-MM solver, and the solver converged at iteration number 11 for both the constrained and unconstrained logarithmic ML-MM solver. The convergence of the cost functions is shown in Fig. 7. One can see the effect of applying the real mode shapes constraint on the value of the initial error. Then, the ML-MM solver reduces the error during its successful iterations. Indeed, the final cost function we obtained after convergence with the constrained ML-MM is still higher than the one for the unconstrained case. Basically, this is due to the fact that the original complex mode shapes exhibit a high level of complexity since the system under test incorporates relatively high level of damping. This is confirmed by the values of the mean phase deviation (MPD) calculated and shown in Table 3 for the estimated complex modes. It can be seen from this table that some complex modes show a rather high MPD. So, the high level of complexity of the
original complex mode shapes limits the capability of the constrained ML-MM solver to reduce more the error between the measured data and the identified modal model.

As a global indicator for the quality of the identified modal model, the mean fitting error and correlation between the measured and the synthesized FRFs are shown in Table 2 for the unconstrained and constrained cases. In general, it can be seen that the quality of the identified modal model has been improved in comparison to the initial fit. This is graphically represented in Fig. 8 by comparing some typical synthesized FRFs with the measured ones. One can see that the modal model obtained by the ML-MM solver closely fits the measurements in comparison to the initial modal model obtained by the linear least-squares solution in the LSFD step. In Table 3, the resonance frequencies, damping ratios, MPD for the complex and real modes are presented. For both cases (i.e. real and complex modes), the ML-MM iterations start from the same starting values with a fixed number of modes and then the modes are ordered according to the frequency values. In the same table the modal assurance criterion (MAC) between the real and complex modes is presented as well. One can see that the frequency estimates for the complex and real modes are very comparable, while the damping estimates are somehow different for some modes. These differences in the damping estimates can be explained by the fact that the proportional

Fig. 5. The 1st fully trimmed-car example: a comparison between the measured and synthesized FRFs corresponding to collocated and orientated sensor–actuator pairs (driving points) where the reciprocity is expected.
damping assumption for such type of structure might be not fully true. The structure is a fully trimmed car that has localized damping (e.g. spot-welds, shock absorbers). So, in such case the damping is not really proportional, and forcing the residue matrices to be purely imaginary could perturb the damping estimates. Even though, estimating modal models that incorporate real mode shapes for such structures (fully trimmed cars) is still an important requirement in the industry for a comparison with finite element model (FEM). The MAC shows that several complex modes are different from the real normal modes. Fig. 9 shows a graphical representation of some estimated real mode shapes.

Fig. 6. The 1st fully trimmed-car example: a comparison between the measured and synthesized data for some typical FRFs that correspond to non-driving points.

Please cite this article as: M. El-Kafafy, et al., Constrained maximum likelihood modal parameter identification applied to structural dynamics, Mech. Syst. Signal Process. (2015), http://dx.doi.org/10.1016/j.ymssp.2015.10.030
5.2. Second fully trimmed car example

The second example that will be used to validate the proposed constrained ML-MM solver is a data set measured for a midsize sedan car. The car has been excited on 8 locations while the acceleration responses have been measured on 192 locations. The added value of this example in comparison to the first trimmed car example is the validation of the constrained ML-MM method with a data set that has more references (e.g., 8 inputs) and higher modal density. In Fig. 10, the geometry of the car is shown, and the locations of the selected references (excitation points) are marked with a green colour. The aim of this modal test was to obtain an experimental modal model that will be used for FEM updating and structural modification purposes. So, obtaining an accurate reciprocal modal model that incorporates real mode shapes was an important requirement. For the confidentiality reasons, the frequency axis in all the shown figures will be made invisible and the resonance frequencies will not be shown.

First, only the reciprocity constraint will be applied on the identified modal model (i.e., reciprocal modal model but with complex-valued mode shape coefficients). Therefore, the ML-MM solver will identify a modal model represented by Eq. (20) by achieving the steps listed and detailed in Section 4.1. Polymax estimator together with the classical LSFD estimator is used to generate starting values for all the parameters of Eq. (20). The stabilization chart constructed by Polymax for this data set is shown in Fig. 11 which shows that there are more than 60 modes within the analysis band.

Table 1
Mean fitting error and mean fitting correlation between the measured and synthesized FRFs for the LSFD and ML-MM with and without applying the reciprocity constraint (1st fully trimmed-car example).

<table>
<thead>
<tr>
<th></th>
<th>Classical LSFD (linear least-squares-based optimization)</th>
<th>ML-MM (Nonlinear least-squares-based optimization)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Unconstrained</td>
<td>Constrained</td>
</tr>
<tr>
<td>Mean fitting error %</td>
<td>4.54</td>
<td>11.13</td>
</tr>
<tr>
<td>Mean fitting correlation %</td>
<td>97.007</td>
<td>95.94</td>
</tr>
</tbody>
</table>

Fig. 7. The 1st fully trimmed-car example: decreasing of the error during the iterations for both the unconstrained and real-modes constrained ML-MM cost functions.

Table 2
Mean fitting error and mean fitting correlation between the measured and synthesized FRFs for the LSFD and Logarithmic ML-MM when applying the real mode shapes constraint (1st fully trimmed-car example).

<table>
<thead>
<tr>
<th></th>
<th>Classical constrained LSFD (linear least-squares-based optimization)</th>
<th>Constrained Logarithmic ML-MM (Nonlinear least-squares-based optimization)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Unconstrained</td>
<td>Constrained</td>
</tr>
<tr>
<td>Mean fitting error %</td>
<td>4.54</td>
<td>10.08</td>
</tr>
<tr>
<td>Mean fitting correlation %</td>
<td>97.007</td>
<td>93.79</td>
</tr>
</tbody>
</table>

5.2. Second fully trimmed car example

The second example that will be used to validate the proposed constrained ML-MM solver is a data set measured for a midsize sedan car. The car has been excited on 8 locations while the acceleration responses have been measured on 192 locations. The added value of this example in comparison to the first trimmed car example is the validation of the constrained ML-MM method with a data set that has more references (e.g., 8 inputs) and higher modal density. In Fig. 10, the geometry of the car is shown, and the locations of the selected references (excitation points) are marked with a green colour. The aim of this modal test was to obtain an experimental modal model that will be used for FEM updating and structural modification purposes. So, obtaining an accurate reciprocal modal model that incorporates real mode shapes was an important requirement. For the confidentiality reasons, the frequency axis in all the shown figures will be made invisible and the resonance frequencies will not be shown.

First, only the reciprocity constraint will be applied on the identified modal model (i.e. reciprocal modal model but with complex-valued mode shape coefficients). Therefore, the ML-MM solver will identify a modal model represented by Eq. (20) by achieving the steps listed and detailed in Section 4.1. Polymax estimator together with the classical LSFD estimator is used to generate starting values for all the parameters of Eq. (20). The stabilization chart constructed by Polymax for this data set is shown in Fig. 11 which shows that there are more than 60 modes within the analysis band.

Then, the constrained ML-MM solver has been applied to the 1536 FRFs. In Fig. 12, the decreasing of the ML-MM cost function for the constrained and unconstrained cases is shown while the mean fitting error and correlation between the measured and synthesized FRFs are shown in Table 4. For the case when the constraint is applied, the initial value of the cost function shown in Fig. 12 and the fitting quality criteria presented in Table 1 show that the classical constrained LSFD estimator, which based on a linear least-squares solution, delivers a very poor quality modal model. Further optimizing this
A modal model using the constrained ML-MM solver significantly improves its quality. Moreover, Table 4 shows that the error between the measured and synthesized FRFs decreases with the number of iterations within the constrained ML-MM solver. Likewise, the correlation between measured and synthesized FRFs increases.

The capability of the obtained modal models to re-synthesize the measured FRFs is graphically represented in Figs. 13 and 14 for some typical reciprocal and non-reciprocal FRFs respectively. From these figures, one can see clearly that the reciprocal modal model obtained by the ML-MM solver more closely fits the measured data.

One more constraint, which is the estimation of real mode shapes, will be applied in addition to the reciprocity constraint which makes the problem a bit harder. In this case, the ML-MM solver will optimize the modal model described in Eq. (25) according to the procedure detailed in Section 4.1. The decrease of the ML-MM cost function during the iterations for the unconstrained and constrained cases is shown in Fig. 15. Comparing the initial value of the ML-MM cost function in Figs. 12 and 15 for the constrained case shows that applying the real mode shapes estimation constraint in addition to the

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**Fig. 8.** The 1st fully trimmed-car example: a comparison between the measured and synthesized data for some typical FRFs applying the real mode shapes constraint.

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reciprocity constraint decreases more the capability of the classical LSFD estimator of giving an accurate modal model. This becomes more obvious when looking at the quality of the fit between the measured and synthesized FRFs which is presented quantitatively in Table 5 and graphically in Figs. 16 and 17. There, it can be seen that applying the real mode shapes

<table>
<thead>
<tr>
<th>Complex modes</th>
<th>Real modes</th>
<th>MAC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Freq. (Hz)</td>
<td>Freq. (Hz)</td>
<td></td>
</tr>
<tr>
<td>Damp. (%)</td>
<td>Damp (%)</td>
<td></td>
</tr>
<tr>
<td>MPD (deg.)</td>
<td>MPD (deg.)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3.92</td>
<td>3.85</td>
</tr>
<tr>
<td>2</td>
<td>4.21</td>
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<td>4.60</td>
</tr>
<tr>
<td>4</td>
<td>6.00</td>
<td>6.02</td>
</tr>
<tr>
<td>5</td>
<td>8.58</td>
<td>8.50</td>
</tr>
<tr>
<td>6</td>
<td>14.56</td>
<td>14.44</td>
</tr>
<tr>
<td>7</td>
<td>15.82</td>
<td>15.84</td>
</tr>
<tr>
<td>8</td>
<td>17.06</td>
<td>16.84</td>
</tr>
<tr>
<td>9</td>
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<td>22.42</td>
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<tr>
<td>15</td>
<td>25.08</td>
<td>24.84</td>
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<tr>
<td>16</td>
<td>25.09</td>
<td>24.88</td>
</tr>
<tr>
<td>17</td>
<td>26.12</td>
<td>26.55</td>
</tr>
<tr>
<td>18</td>
<td>27.13</td>
<td>27.17</td>
</tr>
</tbody>
</table>

Mode 6: Vertical front differential mode

Mode 10: Powertrain vertical coupled to body bending

Mode 13: Body bending coupled to powertrain bending (in phase)

Mode 18: Body bending coupled to powertrain bending (out of phase)

Fig. 9. The 1st fully trimmed-car example: a graphical representation of some typical estimated real mode shapes.

Fig. 10. The 2nd fully trimmed-car example: the geometry of the car under test. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

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constraint together with the reciprocity constraint in the classical constrained LSFD estimator leads to dramatically
degradation on the quality of the fit between the synthesized and measured FRFs. From the same table and figures, it can be
seen that the constrained ML-MM solver outperforms the classical constrained LSFD estimator in terms of the accuracy of
the obtained modal model.

6. Conclusion

A constrained maximum likelihood-based estimation method to establish directly modal models of structural dynamic
systems satisfying desired motivated constraints has been presented. The presented method is the further development of a
recently introduced modal parameter estimation technique (called ML-MM) with the aim to obtain modal models that
satisfy the reciprocity and real mode shapes constraints. In terms of the effectiveness of the proposed method, it was shown
that the method was successfully applied to real experimental data measured from two different fully trimmed cars, and it
was shown to outperform the classical LSFD method when applying those two constraints on the identified modal model.
Fig. 13. The 2nd fully trimmed-car example: Reciprocal frequency response functions of different collocated and orientated sensor–actuator pairs (driving points) where the reciprocity is expected. Black and red, measured reciprocal FRFs; green and blue, synthesized reciprocal FRFs (please note that the green and blue line are exactly the same). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 14. The 2nd fully trimmed-car example: the quality of the fit for some other non-reciprocal FRFs when applying the reciprocity constraint.

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Fig. 15. The 2nd fully trimmed-car example: Unconstrained and constrained ML-MM cost function at different iterations in case of applying both real mode shapes and reciprocity constraints.

Table 5
Mean fitting error and mean fitting correlation between the measured and synthesized FRFs for the LSFD and ML-MM when applying both real mode shapes and reciprocity constraints together (2nd fully trimmed-car example).

<table>
<thead>
<tr>
<th></th>
<th>Classical constrained LSFD (Linear least-squares–based optimization)</th>
<th>Constrained ML-MM (Nonlinear least-squares-based optimization)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean fitting error %</td>
<td>60.94</td>
<td>7.83</td>
</tr>
<tr>
<td>Mean fitting correlation %</td>
<td>66.71</td>
<td>93.33</td>
</tr>
</tbody>
</table>

Fig. 16. The 2nd fully trimmed-car example: A comparison between the measured and the synthesized FRFs when applying both the reciprocity and the real mode shape constraints (driving points FRFs). Black and red, measured reciprocal FRFs; green and blue, synthesized reciprocal FRFs (please note that the blue and the green lines are exactly the same). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

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Acknowledgements

The financial support of the IWT (Flemish Agency for Innovation by science and Technology) (145039) and Siemens Industry Software, through the Innovation mandate IWT project 145039, is gratefully acknowledged.

Appendix A

- Matrix $A_Q$

For reciprocity constraint, assume a modal model that has a summation of $N_m$ modes plus the lower and upper residual terms (Eq. (20}). For each mode, there is a pole $\lambda$ and a participation factors vector $L = [L_1 \ L_2 \ \cdots \ L_{N_i}]$. The matrix $A_Q$ utilized in Eq. (22) is written as follows:

$$A_Q = \begin{bmatrix}
\frac{l_{11} l_{11}}{s_1 - \lambda} + \frac{l_{11} l_{11}}{s_1 - \lambda} & j\left(\frac{l_{11} l_{11}}{s_1 - \lambda} - \frac{l_{11} l_{11}}{s_1 - \lambda}\right) & \cdots & j\left(\frac{l_{11} l_{11}}{s_1 - \lambda} - \frac{l_{11} l_{11}}{s_1 - \lambda}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{l_{11} l_{11}}{s_N - \lambda} + \frac{l_{11} l_{11}}{s_N - \lambda} & j\left(\frac{l_{11} l_{11}}{s_N - \lambda} - \frac{l_{11} l_{11}}{s_N - \lambda}\right) & \cdots & j\left(\frac{l_{11} l_{11}}{s_N - \lambda} - \frac{l_{11} l_{11}}{s_N - \lambda}\right) \\
\frac{l_{11} l_{11}}{s_{N_m} - \lambda} + \frac{l_{11} l_{11}}{s_{N_m} - \lambda} & j\left(\frac{l_{11} l_{11}}{s_{N_m} - \lambda} - \frac{l_{11} l_{11}}{s_{N_m} - \lambda}\right) & \cdots & j\left(\frac{l_{11} l_{11}}{s_{N_m} - \lambda} - \frac{l_{11} l_{11}}{s_{N_m} - \lambda}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{l_{11} l_{11}}{s_1 - \lambda} + \frac{l_{11} l_{11}}{s_1 - \lambda} & j\left(\frac{l_{11} l_{11}}{s_1 - \lambda} - \frac{l_{11} l_{11}}{s_1 - \lambda}\right) & \cdots & j\left(\frac{l_{11} l_{11}}{s_1 - \lambda} - \frac{l_{11} l_{11}}{s_1 - \lambda}\right) \\
\frac{l_{11} l_{11}}{s_{N_m} - \lambda} + \frac{l_{11} l_{11}}{s_{N_m} - \lambda} & j\left(\frac{l_{11} l_{11}}{s_{N_m} - \lambda} - \frac{l_{11} l_{11}}{s_{N_m} - \lambda}\right) & \cdots & j\left(\frac{l_{11} l_{11}}{s_{N_m} - \lambda} - \frac{l_{11} l_{11}}{s_{N_m} - \lambda}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{l_{11} l_{11}}{s_1 - \lambda} + \frac{l_{11} l_{11}}{s_1 - \lambda} & j\left(\frac{l_{11} l_{11}}{s_1 - \lambda} - \frac{l_{11} l_{11}}{s_1 - \lambda}\right) & \cdots & j\left(\frac{l_{11} l_{11}}{s_1 - \lambda} - \frac{l_{11} l_{11}}{s_1 - \lambda}\right) \\
\frac{l_{11} l_{11}}{s_{N_m} - \lambda} + \frac{l_{11} l_{11}}{s_{N_m} - \lambda} & j\left(\frac{l_{11} l_{11}}{s_{N_m} - \lambda} - \frac{l_{11} l_{11}}{s_{N_m} - \lambda}\right) & \cdots & j\left(\frac{l_{11} l_{11}}{s_{N_m} - \lambda} - \frac{l_{11} l_{11}}{s_{N_m} - \lambda}\right)
\end{bmatrix}_{N_m \times N_m}$$

(A.1)

Then, matrix $A_Q$ is transformed to be a real-valued matrix by stacking the real and imaginary parts:

$$A_Q = \begin{bmatrix}
Re(A_Q) \\
Im(A_Q)
\end{bmatrix}$$

(A.2)

In case the estimation of the real mode shapes constraint is applied together with the reciprocity constraint, the scaling factor $Q_r$ is taken as an imaginary quantity, and hence the number of the columns in the matrix $A_Q$ (Eq. (A.1)) has to be halved by omitting the columns which correspond to the real parts.

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6.1. **Matrix \( A_{UR} \)**

When the reciprocity constraint is applied, the partition of the lower and upper residual matrices that corresponds to collocated DOFs forms a reciprocal square matrix with a size \( N_f \times N_f \). So, the total number of the elements per each matrix is \( N_f^2 \), but the number of the elements that have to be estimated per each matrix is \( \frac{N_f(N_f+1)}{2} \) (i.e. the diagonal elements plus half the number of the off-diagonal elements). Therefore, assuming displacement FRFs, the entries of the matrix

\[
A_{UR} = \begin{bmatrix}
[1_{N_f}]_{1,1} & [0_{N_f}] & [0_{N_f}] & [0_{N_f}] & [0_{N_f}] & [0_{N_f}] & \ldots & \ldots & [0_{N_f}] & [0_{N_f}] \\
[0_{N_f}] & [1_{N_f}]_{1,2} & [0_{N_f}] & [0_{N_f}] & [0_{N_f}] & [0_{N_f}] & \ldots & \ldots & [0_{N_f}] & [0_{N_f}] \\
[0_{N_f}] & [0_{N_f}] & [1_{N_f}]_{1,3} & [0_{N_f}] & [0_{N_f}] & [0_{N_f}] & \ldots & \ldots & [0_{N_f}] & [0_{N_f}] \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ldots & \ldots & \vdots & \vdots \\
[0_{N_f}] & [0_{N_f}] & [0_{N_f}] & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
[0_{N_f}] & [1_{N_f}]_{1,2} & [0_{N_f}] & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
[0_{N_f}] & [0_{N_f}] & [1_{N_f}]_{1,3} & [0_{N_f}] & [0_{N_f}] & [0_{N_f}] & \ldots & \ldots & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \ldots & \ldots & \ldots \\
[0_{N_f}] & [0_{N_f}] & [0_{N_f}] & [0_{N_f}] & [0_{N_f}] & [0_{N_f}] & \ldots & \ldots & [0_{N_f}] & [0_{N_f}] \\
[0_{N_f}] & [0_{N_f}] & [0_{N_f}] & [0_{N_f}] & [0_{N_f}] & [0_{N_f}] & \ldots & \ldots & [0_{N_f}] & [1_{N_f}]_{N_f,N_f} \\
\end{bmatrix}_{N_f^2 \times N_f(N_f+1)/2}
\]

(3.3)

where \([1_{N_f}]_{l,k}\) and \([0_{N_f}]_{l,k}\) are all-ones and all-zeros column vectors respectively of a length \( N_f \) with \( l,k = 1,2, \ldots, N_f \). The submatrix \( A_{UR} \) will be written in the same structure as \( A_{UR} \) but with replacing the all-ones column vectors by \([\frac{1}{\pi} \frac{1}{\pi} \ldots \frac{1}{\pi}]_{l,k}^T\) with \( s_k = j\omega_k \).

**Transformation matrix \( T \)**

In Eq. (23), a vector containing the full elements of the reciprocal residual matrices are obtained as the product of the multiplication of a transformation matrix \( T \) and the vector that contains the estimated \( \frac{N_f(N_f+1)}{2} \) residual elements. This transformation matrix is written as follows,

\[
T = \begin{bmatrix}
1_{1,1} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1_{1,2} & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1_{1,3} & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 1_{1,N_f} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1_{2,1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1_{2,2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{3,3} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{N_f,N_f} \\
\end{bmatrix}_{N_f^2 \times N_f(N_f+1)/2}
\]

(4.4)

**Appendix B: mean fitting error and mean fitting correlation**

In the validation and discussion sections, the quality of the fit between the synthesized and measured FRFs has been evaluated by means of so-called mean fitting error and mean fitting correlation. The mean fitting error and mean fitting correlation
correlation are calculated according to the following equations:

\[ E \% = 100 \times \frac{1}{N_o N_i} \sum_{o=1}^{N_o} \sum_{i=1}^{N_i} \sum_{k=1}^{N_f} \frac{|\hat{H}_{oi}(\omega_k) - H_{oi}(\omega_k)|^2}{\sum_{k=1}^{N_f} |\hat{H}_{oi}(\omega_k)|^2} \]  

(B.1)

\[ C \% = 100 \times \frac{1}{N_o N_i} \sum_{o=1}^{N_o} \sum_{i=1}^{N_i} \sum_{k=1}^{N_f} \frac{|\hat{H}_{oi}(\omega_k)H_{oi}^*(\omega_k)|^2}{(\sum_{k=1}^{N_f} |\hat{H}_{oi}(\omega_k)|^2)(\sum_{k=1}^{N_f} |H_{oi}(\omega_k)|^2)} \]  

(B.2)

with \( \hat{H}_{oi}(\omega_k) \) and \( H_{oi}(\omega_k) \) the synthesized and the measured FRFs respectively for output \( o \) and input \( i \), \( N_o \) the number of outputs, and \( N_i \) the number of inputs. \((\cdot)^*\) stands for the complex conjugate of a complex number

**Appendix C : Matrix \([\hat{H}]\)**

Consider a frequency response functions matrix \([FRFs]_k \subset \mathbb{C}^{N_o \times N_i}\) with \( N_o \) outputs, \( N_i \) inputs and \( k = 1, 2, ..., N_f \) where \( N_f \) is the number of frequency lines, the matrix \([\hat{H}] \in \mathbb{R}^{N_o \times 2N_f N_i}\) used in Eq. (13) is given as follows:

\[
[\hat{H}] = \begin{bmatrix} [FRFs_{11}] & [FRFs_{12}] & \cdots & [FRFs_{1N_f}] \\ [FRFs_{21}] & [FRFs_{22}] & \cdots & [FRFs_{2N_f}] \\ \vdots & \vdots & \ddots & \vdots \\ [FRFs_{N_i1}] & [FRFs_{N_i2}] & \cdots & [FRFs_{N_iN_f}] \end{bmatrix}  
\]

(C.1)

**References**


[22] I. Rooney, J. Buck, K. Wage, Implementing physical constraints for noise only normal mode shape estimation. in Meeting on Acoustics (ICA 2013), Montreal, Canada, 20123, pp. 1–8.


